OPERATOR AND INDICATOR METHODS IN ORDER STATISTICS

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Dedicated to

My Parents, My Brothers and Sister, My Supervisor
and

Sir Syed Ahmad Khan

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In the Name of Allah, the Most Beneficent and Merciful

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Contents

Preface

The present dissertation entitled "***Operator and Indicator Methods in Order Statistics***" is a brief collection of the work done so far on the subject. I have tried my best to include sufficient and relevant materials in the systematic way, which are contained in five chapters.

Chapter 1, is of introductory nature in which some concepts, which may be helpful to grasp the ideas contained in the remaining chapters are discussed.

In chapter 2, various recurrence relations for the single moments of order statistics are considered. Recurrence relations for single moments of order statistics for some general form of distributions are also discussed.

In chapter 3, recurrence relations for the product moments of order statistics are considered. Here also the recurrence relations for the product moments of order statistics for some general form of distributions are discussed.

In chapter 4, recurrence relations and identities of order statistics from independent and non- identically distributed random variables are discussed.

Finally in the last chapter, we have discussed operator and indicator methods used for the derivation of various recurrence relations and identities.

In the end, a comprehensive list of references referred into this dissertation is given.

Chapter 1

Introduction

1.Order Statistics:

When the independent and identically distributed (*iid*) random variables $X_1, X_2, \cdots X_n$, from a distribution having cumulative distribution function $F(x)$ and probability density function $f(x)$, are arranged in order of magnitude and written as

$$X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$$

then we call $X_{r:n}$ the rth order statistics $(r = 1,2,\cdots,n)$.Note that $X_{r:n}'s$ are necessarily dependent because of order relations among them.

1.1 Cumulative Distribution Function (*cdf*) of a Single Order Statistics:

Let $F_{r:n}(x),\ r = 1,2,\cdots n$ denote the *cdf* of rth order statistics $X_{r:n}$, then

$$F_{r:n}(x) = P(X_{r:n} \le x)$$

$$= P(at\,least\,r\,of\,the\,X_i's\,are\,less\,than\,or\,equual\,to\,x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i} \qquad (1.1)$$

by using binomial probability model.

An alternative form of (1.1) is

$$F_{r:n}(x) = F^r(x) \sum_{j=0}^{n-r} \binom{r+j-i}{r-i} [1-F(x)]^j, \tag{1.2}$$

where the right hand side is the sum of the probabilities of exactly r of $X_1, X_2, \cdots, X_{r+j}$, including X_{r+j}, are less than or equal to x.

Also from the well-known relation between binomial sums and the incomplete beta function, we have

$$F_{r:n}(x) = I_{F(x)}(r, n-r+1), \tag{1.3}$$

where

$$I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1}\,dt, \quad B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

is the incomplete beta funcrion.Thus $F_{r:n}(x)$ can also be evaluated from tables of $I_p(a,b)$.

Taking $r = 1$ *and* n in (1.1), we have respectively

$$F_{1:n}(x) = 1 - [1 - F(x)]^n$$

and

$$F_n(x) = [F(x)]^n.$$

These results are valid for both discrete and continuous distributions.

1.2 Probability Density Function (*pdf*) of a Single Order Statistics:

If $X_i's$ are *iid* continuous random variables, then the *pdf* of rth order statistics is given as

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \tag{1.4}$$

Thus

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x)$$

and

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x)$$

1.3 Joint Probability Density Function of Two Order Statistics:

The joint *pdf* $f_{r,s:n}(x,y), x<y$ of two order statistics $X_{r:n}$ and $X_{s:n}$ where $r<s$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$\times [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x) f(y) \tag{1.5}$$

Thus the joint *pdf* of $1st$ and nth order statistics is

$$f_{1,n:n}(x,y) = \frac{n!}{(n-2)!}[F(y)-F(x)]^{n-2} f(x) f(y). \tag{1.6}$$

1.4 Joint Cumulative Distribution Function of Two Order Statistics:

The joint *cdf* $F_{r,s:n}(x,y)$ $x<y$ of two order statistics $X_{r:n}$ and $X_{s:n}$ may be obtained by integrating the joint *pdf* $f_{r,s:n}(x,y)$ as well as by a direct argument valid also in discrete case. For $x<y,$ we have

$$F_{r,s:n}(x,y) = P(at\,least\,r\,X_i\text{'s} \le x, at\,least\,s\,X_j\text{'s} \le y)$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} P(exactly\, i\, X_i\text{'}s \le x, exactly\, j\, X_j\text{'}s \le y)$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^i [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j}$$

By using the identity given by Arnold *et al.* (1992), we have $\sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^i [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j}$

$$= \int_0^{F(x)} \int_u^{F(y)} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} dv du$$

$$0 \le F(x) \le F(y) \le 1$$

Thus

$$F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$\times \int_0^{F(x)} \int_u^{F(y)} u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} dv du \qquad (1.7)$$

$$= I_{F(x).F(Y)}(r, s-r, n-s+1), \ -\infty < x, y < \infty \qquad (1.8)$$

which is incomplete bivariate beta function.

It may be noted that for $x \ge y$, the inequality $X_{s:n} \le y$ implies $X_{r:n} \le x$, so that

$$F_{r,s:n}(x,y) = F_s(y)$$

1.5 Joint Probability Density Function of More Than Two Order Statistics:

The joint *pdf* of k order statistics $X_{r_1}, X_{r_2}, \cdots, X_{r_k}$, where $1 \le r_1 \le r_2 \le \cdots \le r_k \le n$ and $1 \le k \le n$, is, for $x_1 \le x_2 \le \cdots \le x_k$, given by

$$f_{r_1, r_2, \cdots, r_k}(x_1, x_2, \cdots, x_k) = \frac{n!}{(r_1 - 1)!(r_2 - r_1 - 1)! \cdots (r_k - r_{k-1} - 1)!(n - r_k)!}$$

$$\times [F(x_1)]^{r_1 - 1} f(x_1) [F(x_2) - F(x_1)]^{r_2 - r_1 - 1} f(x_2) \cdots$$

$$\times [F(x_k) - F(x_{k-1})]^{r_k - r_{k-1} - 1} f(x_k) [1 - F(x_k)]^{n - r_k} \tag{1.9}$$

In particular, the joint *pdf* of all the n order statistics is obtained on taking $k = n$ in (1.9).

Hence the joint *pdf* of $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ is given by

$$f_{1,2,\cdots,n:n}(x_1, x_2, \cdots, x_n) = n! f(x_1) f(x_2) \cdots f(x_n) \tag{1.10}$$

1.6. Conditional Distribution of Order Statistics:

Result: The conditional *pdf* of $X_{s:n}$ given $X_{r:n} = x, (r < s)$ is

$$\frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{\{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} f(y)}{\{1 - F(x)\}^{n-r}}, \; x \le y$$

which is just the unconditional *pdf* of the $(s-r)th$ order statistics in a sample of size $(n-r)$ drawn from $\frac{f(y)}{1-F(x)}, y \ge x,$ that is from the parent distribution truncated on the left at x. Also, the conditional distribution of $X_{r:n}$ given $X_{s:n} = y, (r < s)$ is

$$\frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{\{F(y)-F(x)\}^{s-r-1}\{F(x)\}^{r-1}f(x)}{\{F(y)\}^{s-1}}, x \le y$$

which is unconditional *pdf* of the r *th* order statistics in a sample of size $(s-1)$ truncated on the right at y.

2. Indicator Function:

Let Ω be the sample space and $A \subseteq \Omega$, then a function defined as

$$\chi_A = \begin{cases} 1 & when\ A\ occurs \\ 0 & when\ A\ does\ not\ occur \end{cases}$$

is called the indicator function of the set A .

3. Operators:

3.1. Difference Operator:

Let us define a function $y = f_x$ where x is an independent variable and $y = f_x$ is a dependent variable. Suppose we are given equidistant values (finite in number) $a, a+h, a+2h, \cdots$ of the variable x at an interval of 'h'.

Then the corresponding values of the interval $y = f_x$ are $f_a, f_{a+h}, f_{a+2h}, \cdots$

The values of the independent variables x are known as argument and the corresponding values of the dependent variable y are called entries.

3.2. Forward Difference:

Forward difference operator Δ is defined as

$\Delta f_x = f_{x+h} - f_a$; $x = a, a+h, a+2h, \ldots$

Interval h is known as the interval of difference. Thus

$\Delta f_x = f_{a+h} - f_a,$

$$\Delta f_{a+h} = f_{a+h+h} - f_{a+h},$$

so on.

Δf_x's are known as first order differences.

$$\Delta^2 f_x = \Delta(\Delta f_x) = \Delta(f_{x+h} - f_x)$$

$$= \Delta f_{x+h} - \Delta f_x,\ x = a, a+h, a+2h, \ldots$$

Similarly all the higher order differences are obtained.

The backward differences denoted by ∇, is defined as

$$\nabla f_{x+h} = f_{x+h} - f_x = \Delta f_x;\ x = a, a+h, a+2h, \ldots$$

Thus the backward differences of f_{x+h} is same as the forward difference of f_x.

3.3. The Shift Operator:

The shift operator E is defined as

$$E f_x = f_{x+h}$$

i.e., it results in increasing the argument by the interval of difference.

$$E^2 f_x = E(E f_x) = E(f_{x+h}) = f_{x+2h}$$

In general

$$E^r f_x = f_{x+rh}$$

3.4. Various Relations among Δ, ∇, E, and Differential Operator D:

(i) $1 + \Delta = E \Rightarrow \Delta = E - 1$

(ii) $1 - \nabla = E^{-1} \Rightarrow E = (1 - \Delta)^{-1}$

(iii) $E = 1 + \Delta = e^{hD}$

3.5. Properties of E and D:

(i) The operators E and D are commutative in operation w.r.t constant i.e.

$\Delta(Cf_x) = C\Delta f_x$ and $E(Cf_x) = CEf_x$

(ii) The operators Δ and E are distributive w.r.t addition, i.e,

$\Delta(f_x + g_x) = \Delta f_x + \Delta g_x$ and

$E(f_x + g_x) = E f_x + E g_x$

(iii) The operators Δ and E are linear, i.e, if a and b are constants, then

$\Delta(af_x + bg_x) = a\Delta f_x + b\Delta f_y$ and

$E(af_x + bg_x) = aE f_x + bE f_y$

(iv) The operators Δ and E obey the law of indices, i.e.,

$\Delta^m \Delta^n f_x = \Delta^{m+n} f_x$ and

$E^m E^n f_x = E^{m+n} f_x$

Remarks: (i) The operators Δ and E are not commutative w.r.t a variable i.e,

$\Delta(f_x g_x) \neq f_x \Delta g_x$, and

$E(f_x g_x) \neq f_x E g_x$

(ii) $\Delta \frac{f_x}{g_x} = \frac{g_x \Delta f_x - f_x \Delta g_x}{g_x g_{x+h}}$

(iii) $\Delta(constt.) = 0$

(iv) $\Delta^n f_x = (E-1)^n f_x$

$$= \left[\sum_{r=0}^{n} \binom{n}{r} E^r (-1)^{n-r} \right] f_x$$

$$= \sum_{r=0}^{n} \left[(-1)^{n-r} \binom{n}{r} f_{x+rh} \right]$$

3.6. Fundamental Theorem of Finite Differences:

If f_x is a polynomial of nth degree in x, then

$$\Delta^r f_x = \begin{cases} constt. & if\, r = n \\ 0 & if\, r > n \end{cases}$$

In other words, the nth order difference of a polynomial of nth degree is constant and higher differences are zero.

4. Truncated Distribution:

4.1. Doubly Truncated Distribution:

Let X be a random variable with *pdf* $f_1(x)$ and *cdf* $F_1(x)$. If for given P_1 and Q_1

$$\int_{-\infty}^{Q_1} f_1(x)dx = Q \text{ and } \int_{-\infty}^{P_1} f_1(x)dx = P. \tag{4.1}$$

Then the doubly truncated *pdf* is given by

$$f(x) = \frac{f_1(x)}{P-Q}; \qquad x \in (Q_1, P_1) \tag{4.2}$$

and the corresponding *df* by

$$F(x) = \frac{F_1(x)-Q}{P-Q}; \qquad x \in (Q_1, P_1) \tag{4.3}$$

Remark. For non- truncated case we put $P=1$ and $Q=0$.

4.2. Left Truncated Distribution:

If the values below a certain limit, x, are not observed, the distribution is said to be truncated on left.

Putting $Q = F_1(x)$ and $P=1$, in equation (4.2) the left truncated distribution has *pdf*,

$$f(t) = \frac{f_1(t)}{1-F_1(x)}; \qquad x < t < \infty \tag{4.4}$$

4.3. Right Truncated Distribution:

If the values larger than an upper limit , y, are not observed, the distribution is said to be truncated on right.

Putting $P = F_1(y)$ and $Q=0$, in equation (4.2) the right truncated distribution has *pdf*

$$f(t) = \frac{f_1(t)}{F_1(y)}, \quad -\infty < t < y \tag{4.5}$$

5. Some Distributions:

(a) Weibull Distribution:

$$f_1(x) = p\alpha^{-1}\left(\frac{x-\mu}{\alpha}\right)^{p-1} \exp\left[-\left(\frac{x-\mu}{\alpha}\right)^{p}\right]; \quad 0 \le x \le a, \quad a, v > 0 \quad (5.1)$$

$$F_1(x) = 1 - \exp\left[\left\{-\alpha^{-1}\left(\frac{x-\mu}{\alpha}\right)^{p}\right\}\right], \; 0 \le x \le a, \quad a, v > 0 \quad (5.2)$$

The *pdf* of standard Wiebull distribution $(\alpha = 1, \mu = 0)$ is given by

$g_1(x) = px^{p-1}e^{-x^p}; \quad x > 0, p > 0$ and the corresponding *df* is given by

$$G_1(x) = 1 - e^{-x^p}$$

Moments of standard Weibull distribution are given by

$$E(X^n) = \alpha^{n/p}\Gamma\left(1 + \frac{n}{p}\right),$$

If we put $p = 1$ and $\mu = 0$ in (5.1), it gives the *pdf* of exponential distribution.

(b) Power Function Distribution:

$$f_1(x) = va^{-v}x^{v-1}, \quad 0 \le x \le a, \quad a, v > 0 \quad (5.3)$$

$$F_1(x) = a^{-v}x^{v}, \quad 0 \le x \le a, \quad a, v > 0 \quad (5.4)$$

$$E(X^n) = \frac{va^{-v}a^{n}}{n+v}$$

(c) Pareto Distribution:

$$f_1(x) = va^{v}x^{-v-1}, \quad x > 0, \, a, v > 0 \quad (5.5)$$

$$F_1(x) = a^{v}(1 - x^{-v}), \quad x > 0, \, a, v > 0 \quad (5.6)$$

$$E(X^n) = \frac{va^v}{v-n}$$

(d) Cauchy Distribution:

$$f_1(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty \tag{5.7}$$

$$F_1(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x, \quad -\infty < x < \infty \tag{5.8}$$

For Cauchy distribution the moments of of order <1 exists, but the moments of order ≥ 1 do not exist.

(e) Gamma Distribution:

$$f_1(x) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}, \qquad 0 < x < \infty, \ \alpha, p > 0 \tag{5.9}$$

$$F_1(x) = \frac{\alpha^p}{\Gamma(p)} \int_0^x e^{-\alpha x} x^{p-1} dx, \ \ 0 < x < \infty \tag{5.10}$$

The *df* of gamma distribution is called incomplete gamma function.

$$E(X^n) = \frac{\Gamma(p+n)}{\alpha^p \Gamma(p)}$$

(f) Beta Distribution:

$$f_1(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad a,b > 0, 0 < x < 1 \text{ where} \tag{5.11}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$F_1(x) = \int_0^x \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} dt, \quad 0 < x < 1,\ a,b > 0 \tag{5.12}$$

$$E(X^n) = \frac{\Gamma(a+n)\Gamma(a+b)}{\Gamma(a+b+n)\Gamma(a)}$$

(g) Exponential Distribution:

$$f_1(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0 \tag{5.13}$$

$$F_1(x) = 1 - e^{-x/\beta}, \quad x > 0 \tag{5.14}$$

$$E(X^n) = n!\beta^n$$

(h) Burr Distribution:

$$f_1(x) = m p \theta x^{p-1}(1+\theta x^p)^{-(m+1)}, \quad -\infty < x < \infty \tag{5.15}$$

$$F_1(x) = 1 - (1-\theta x^p)(1+\theta x^p)^{-(m+1)}, \quad -\infty < x < \infty \tag{5.16}$$

(i) Logistic Distribution:

$$f_1(x) = \frac{\alpha e^x}{(1+e^x)^{\alpha+1}}, \quad -\infty < x < \infty \tag{5.17}$$

$$F_1(x) = 1 - (1+e^x)^{-\alpha}, \quad -\infty < x < \infty \tag{5.18}$$

(j) Log-logistic Distribution:

$$f_1(x) = \frac{p\theta x^{p-1}}{(1+\theta x^p)^2}, \quad -\infty < x < \infty \tag{5.19}$$

$$F_1(x) = 1 - \frac{1}{1+\theta x^p} \tag{5.20}$$

Chapter 2

Recurrence Relations for the Single Moments of Order Statistics

1. Introduction:

Let $X_1, X_2, X_3, \ldots X_n$ be a random sample from a continuous population with distribution function $F(x)$ and *pdf* $f(x)$. Let $X_{r:n}$ be the *r th* order statistics with *pdf*

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^r [1-F(x)]^{n-r} f(x)$$

then $k\,th$ moment of $X_{r:n}$ is defined as

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k)$$

Sen (1959) has shown that if $E|X|^\delta$ exists for some $\delta>0$, then $\mu_{r:n}^{(k)}$ exists for all r satisfying $r_0 \le r \le n - r_0 + 1$, where $r_0\delta = k$.

2.Recurrence Relations for Moments:

Khan *et al.* (1983a) have obtained the recurrence relations for the single order statistics.

In case of truncation from both the sides

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-1} \{1-F(x)\}^{n-r} f(x) dx \qquad (2.1)$$

Let $g(x)$ be a Borel measurable function of x, then

$$E\{g(X_{r:n})=\int_{Q_1}^{P_1} g(x)f_{r:n}(x)dx, \quad Q_1\le x\le P_1 \tag{2.2}$$

Theorem 2.1 (Khan & Ali, 1997) For $1\le r\le n; n=1,2,\cdots$

$$E\{g(X_{r:n})\}-E\{g(X_{r-1:n-1})\}$$

$$=\binom{n-1}{r-1}\int_{Q_1}^{P_1} g'(x)\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}dx \tag{2.3}$$

Theorem 2.2 (Khan & Ali, 1997) For $1\le r\le n\,; n=1,2,\cdots$

$$E\{g(X_{r:n})\}-E\{g(X_{r-1:n})\}$$

$$=\binom{n}{r-1}\int_{Q_1}^{P_1} g'(x)\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}dx \tag{2.4}$$

Theorem 2.3 (Khan & Ali, 1997) For $1\le r\le n\,; n=1,2,\cdots$

$$E\{g(X_{r-1:n-1})\}-E\{g(X_{r-1:n})\}$$

$$=\binom{n-1}{r-2}\int_{Q_1}^{P_1} g'(x)\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}dx \tag{2.5}$$

It is important to note that all the above Theorems lead to establish the well-known relation (David, 1981),

$$(n-r)E\{g(X_{r:n})\}+rE\{g(X_{r+1:n})\}=nE\{g(X_{r:n-1})\} \tag{2.6}$$

The usual technique used was to express $\{1-F(x)\}$ as a function of x and $f(x)$.

As convention, we write for $n\ge 1$

$$X_{0:n} = Q_1$$
$$X_{n:n-1} = P_1 \tag{2.7}$$

3. Examples:

Khan *et al.* (1983a) have obtained the recurrence relations for the single moments of some well-known distributions.

3.1. Doubly Truncated Weibull and Exponential Distributions: (Khan *et al.*, 1983a)

$$f(x) = \frac{px^{p-1}e^{-x^p}}{P-Q}, \quad -\log(1-Q) \le x^p \le -\log(1-P), \; p > 0 \tag{3.1}$$

Here $Q_1^p = -\log(1-Q), P_1^p = -\log(1-P)$.Let $Q_2 = (1-Q)/(1-P)$ and $P_2 = (1-P)/(P-Q)$, then

$$\begin{aligned}\{1-F(x)\} &= -P_2 + \frac{e^{-x^p}}{P-Q} \\ &= -P_2 + \frac{1}{p}x^{1-p}f(x)\end{aligned} \tag{3.2}$$

Putting the value of $\{1-F(x)\}$, into (2.7), we get

$$\mu_{1:n}^{(k)} - Q_1^k = k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^{n-1}\left\{-P_2 + \frac{1}{p}x^{1-p}f(x)\right\}dx$$

$$= -P_2\left\{\mu_{1:n-1}^{(k)} - Q_1^k\right\} + \frac{k}{np}n\int_{Q_1}^{P_1} x^{k-p}\{1-F(x)\}^{n-1}f(x)dx$$

Thus in view of (2.7)

$$\mu_{1:n}^{(k)} = -P_2\mu_{1:n-1}^{(k)} + P_2Q_1^k - \frac{k}{np}\mu_{1:n}^{(k-p)} + Q_1^k$$

$$= -P_2\mu_{1:n-1}^{(k)} - \frac{k}{np}\mu_{1:n}^{(k-p)} + Q_1^k(1+P_2)$$

$$= -P_2\mu_{1:n-1}^{(k)} - \frac{k}{np}\mu_{1:n}^{(k-p)} + Q_1^k Q_2$$

since $1+P_2 = Q_2$ (3.3)

In view of (2.7), it can be shown that

$$\mu_{1:1}^{(k)} = -P_2 P_1^{(k)} + Q_2 Q_1^k - \frac{k}{p}\mu_{1:1}^{(k-p)} \tag{3.4}$$

For the *rth* order statistics, the relationship is given by (from Theorems 2.2 and 2.3)

$$\mu_{r:n}^{(k)} - \mu_{r-1 n-1}^{(k)} = \binom{n-1}{r-1} k \int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}\left\{-P_2 + \frac{1}{p}x^{1-p}f(x)\right\}dx$$

$$= -P_2\frac{n-1}{n-r}\binom{n-2}{r-1} k \int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}dx$$

$$+\frac{k}{np}\frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} x^{k-p}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x)dx$$

Thus in view of (2.1) and (2.3), we get

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = -P_2\frac{n-1}{n-r}\left\{\mu_{r:n-1}^{(k)} - \mu_{r-1:n-2}^{(k)}\right\} + \frac{k}{np}\mu_{r:n}^{(k-p)}$$

Using the recurrence relation

$$r\mu_{r+1:n}^{(k)} = n\mu_{r:n-1}^{(k)} - (n-1)\mu_{r:n}^{(k)}, \tag{3.5}$$

we get

$$\mu_{r:n}^{(k)} = Q_2\mu_{r-1:n-1}^{(k)} - P_2\mu_{r:n-1}^{(k)} + \frac{k}{np}\mu_{r:n}^{(k-p)} \tag{3.6}$$

For $r = n$, from (2.3) we get

$$\mu_{n:n}^{(k)} = Q_2\mu_{n-1:n-1}^{(k)} - P_2P_1^k + \frac{k}{np}\mu_{n:n}^{(k-p)}. \tag{3.7}$$

The method presented above is the generalization of the results of Balakrishnan & Joshi (1981a).The exact and explicit expressions for the non-truncated Weibull distribution are given by Leiblein (1955).The results given by Joshi (1979a) for doubly truncated exponential distribution are obtained by setting p=1.Also,the results of Joshi (1978) for non-truncated and right truncated exponential distributions are obtained respectively for *p=1* by setting *P=1,Q=0* and *Q=0* in (3.3), (3.4), (3.6) and (3.7). For related results reference may also be made to Saleh *et al.* (1975).

3.2 Doubly Truncated Power Function Distributions: (Khan *et al.*, 1983a)

$$f(x) = \frac{va^{-v}x^{v-1}}{P-Q}, \quad aQ^{1/v} \le x \le aP^{1/v},\ a, v > 0 \tag{3.8}$$

Here, $Q_1 = aQ^{1/v}$, $P_1 = aP^{1/v}$.

Let $P_2 = P/(P-Q)\, and\, Q_2 = Q/(P-Q)$.Then

$$\{1-F(x)\} = P_2 - \frac{x}{v}f(x). \tag{3.9}$$

Thus, from (2.7) and (3.9),

$$\mu_{1:n}^{(k)} - Q_1^k = P_2\left[\mu_{1:n-1}^{(k)} - Q_1^K\right] - \frac{k}{nv}\mu_{1:n}^{(k)}.$$

That is,

$$\mu_{1:n}^{(k)} = \left[P_2\mu_{1:n-1}^{(k)} - Q_2Q_1^k\right]\frac{nv}{nv+k}. \tag{3.10}$$

In view of (2.7),

$$\mu_{1:1}^{(k)} = \left[P_2 P_1^k - Q_2 Q_1^k\right]\frac{v}{v+k}. \tag{3.11}$$

For the *rth* order statistics,

$$\mu_{r;n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \frac{(n-1)P_2}{(n-r)}\left[\mu_{r:n-1}^{(k)} - \mu_{r-1:n-2}^{(k)}\right] - \frac{k}{nv}\mu_{r:n}^{(k)}$$

Using the relation (3.5), one gets

$$\mu_{r:n}^{(k)} = \left[P_2 \mu_{r:n-1}^{(k)} - Q_2 \mu_{r-1:n-1}^{(k)}\right]\frac{nv}{nv+k}. \tag{3.12}$$

Also, for $r = n$, we get

$$\mu_{n:n}^{(k)} = \left[P_2 P_1^k - Q_2 \mu_{n-1:n-1}^{(k)}\right]\frac{nv}{nv+k}. \tag{3.13}$$

These relations were also obtained established by Balakrishnan & Joshi (1981b). For non-truncated case we put *P=1* and *Q=0.*

3.3. Doubly Truncated Pareto Distributions: (Khan *et al.*, 1983a)

$$f(x) = \frac{va^v x^{-v-1}}{P-Q}, \quad a(1-Q)^{-1/v} \le x \le a(1-P)^{-1/v},\ a, v > 0 \tag{3.14}$$

Here, $Q_1 = a(1-Q)^{-1/v}$, $P_1 = a(1-P)^{-1/v}$. Let

$P_2 = (P-1)/(P-Q)\, and\, Q_2 = (Q-1)/(P-Q)$. Then

$$\{1-F(x)\} = \frac{x}{v}f(x) + P_2$$

Thus,

$$\mu_{1:n}^{(k)} - Q_I^k = k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^{n-1}\left\{\frac{x}{v}f(x) + P_2\right\}dx$$

$$=\frac{k}{nv}\mu_{1:n}^{(k)}+P_2\left[\mu_{1:n-1}^{(k)}-Q_1^k\right]$$

or

$$(nv-k)\mu_{1:n}^{(k)}=\left[P_2\mu_{1:n-1}^{(k)}-Q_2Q_1^k\right]nv, \qquad nv\neq k \tag{3.15}$$

In particular,

$$(v-k)\mu_{1:1}^{(k)}=\left[P_2P_1^k-Q_2Q_1^k\right]v\,, \quad v\neq k \tag{3.16}$$

For the rth order statistics, $2\leq r\leq n-1$,

$$\mu_{r:n}^{(k)}-\mu_{r-1:n-1}^{(k)}=\frac{k}{nv}\mu_{r:n}^{(k)}+P_2\left(\frac{n-1}{n-r}\right)\left\{\mu_{r:n-1}^{(k)}-\mu_{r-1:n-2}^{(k)}\right\}.$$

Using the recurrence relation given in (3.5), we get

$$(nv-k)\mu_{r:n}^{(k)}=\left[P_2\mu_{r:n-1}^{(k)}-Q_2\mu_{r-1;n-1}^{(k)}\right]nv, \qquad nv\neq k, \tag{3.17}$$

and for $r=n$, it can be seen that

$$(nv-k)\mu_{n:n}^{(k)}=\left[P_2P_1^k-Q_2Q_1^k\right]nv, \qquad nv\neq k, \tag{3.18}$$

In case $nv=k$, from (3.15), (3.17) and (3.18) we get, respectively,

$$\mu_{1:n-1}^{(k)}=\frac{Q_2}{P_2}Q_1^k \quad n>1, \tag{3.19}$$

$$\mu_{r:n-1}^{(k)}=\frac{Q_2}{P_2}\mu_{r-1:n-1}^{(k)}, \quad 2\leq r\leq n-1, \tag{3.20}$$

$$\mu_{n-1:n-1}^{(k)}=\frac{Q_2}{P_2}P_1^k. \tag{3.21}$$

However, this result may not be used to evaluate $\mu_{1:1}^{(k)}$ and $\mu_{n:n}^{(k)}$, when $nv=k$.For $\mu_{1:1}^{(k)}$, it can be easily seen by direct integration that

$$\mu_{1:1}^{(k)} = \frac{a^{v} \log(Q_2/P_2)}{P-Q}, \qquad v=k. \tag{3.22}$$

These results were obtained by Balakrishnan & Joshi (1982). For non-truncated case $(P=1, Q=0)$ one, may refer to Malik (1966).

3.4 Doubly Truncated Cauchy Distributions: (Khan *et al.*, 1983a)

$$f(x) = \frac{1}{(P-Q)\pi}\frac{1}{1+x^2}, \qquad Q_1 \le x \le P_1, \tag{3.23}$$

where Q_1 and P_1 are obtained by

$$\int_{-\infty}^{Q_1} f(x)dx = Q, and \int_{P_1}^{\infty} f(x)dx = 1-P.$$

Therefore,

$$(P-Q)\pi(1+x^2)f(x) = 1 \tag{3.24}$$

Now in view of (3.24),

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} =$$

$$\binom{n-1}{r-1} k \int_{Q_1}^{P_1} (P-Q)\pi(1+x^2)x^{k-1}[F(x)]^{r-1}[1-F(X)]^{n-r+1} f(x)dx$$

$$= \frac{(n+1-r)}{(n+1)n}\pi(P-Q)\left\{\mu_{r:n+1}^{(k-1)} + \mu_{r:n+1}^{(k+1)}\right\}$$

or

$$\mu_{r:n+1}^{(k+1)} = \frac{n(n+1)}{\pi(P-Q)(n+1-r)k}\left[\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)}\right] - \mu_{r:n+1}^{(k-1)}.$$

Replacing $n+1$ by n and $k+1$ by k, we get

$$\mu_{r:n}^{(k)} = \frac{n(n-1)}{\pi(P-Q)(n-r)(k-1)}\left[\mu_{r:n-1}^{(k-1)} - \mu_{r-1:n-2}^{(k-1)}\right] - \mu_{r:n}^{(k-2)}.$$

Barnett (1966) has given the relation as

$$\mu_{r:n}^{(k)} = \frac{n}{\pi(k-1)}\left[\mu_{r:n-1}^{(k-1)} - \mu_{r-1:n-1}^{(k-1)}\right] - \mu_{r:n}^{(k-2)} \tag{3.25}$$

This could have been obtained by establishing

$$\mu_{r:n}^{(k)} - \mu_{r-1:n}^{(k)} = \binom{n}{r-1} k \int_{Q_1}^{P_1} x^{k-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1} f(x)dx$$

on the lines of Theorem 2.1 and then replacing 1 by

$(P-Q)\pi(1+x^2)f(x)$, we have

$$\mu_{r:n}^{(k)} - \mu_{r-1:n}^{(k)}$$

$$= \binom{n}{r-1} k\pi(P-Q) \int_{Q_1}^{P_1} x^{k-1}(1+x^2)[F(x)]^{r-1}[1-F(x)]^{n-r+1} f(x)dx$$

$$= \frac{k\pi(P-Q)}{n+1}\left[\mu_{r:n+1}^{(k-1)} + \mu_{r:n+1}^{(k+1)}\right] \tag{3.26}$$

Replacing $n+1\, by\, n, k+1\, by\, k,$ and rearranging (3.26), we get (3.25), with $P=1\, and\, Q=1$. Barnett (1966) has tabulated means of order statistics using the relation (3.25).

3.5. Symmetric Truncated Logistic Distributions: (Khan *et al.*, 1983a)

$$f(x) = \frac{e^{-x}}{(P-Q)(1+e^{-x})^2}, \quad \log\left(\frac{Q}{1-Q}\right) \le x \le \log\left(\frac{P}{1-P}\right). \tag{3.27}$$

Here $Q_1 = \log[Q/(1-Q)]\, and\, P_1 = \log[P/(1-P)]$. We have

$$F(x) = \frac{(1+e^{-x})^{-1} - Q}{P-Q},$$

$$1-F(x) = \frac{P-(1+e^{-x})^{-1}}{P-Q}.$$

Thus, in case of symmetric truncation ($Q = 1 - P$), we have

$$F(x)[1 - F(x)] = -\frac{PQ}{(P-Q)^2} + \frac{P+Q}{P-Q} f(x). \tag{3.28}$$

Thus

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} =$$

$$\binom{n-1}{r-1} k \int_{Q_1}^{P_1} x^{k-1} [F(x)]^{r-2} [1 - F(x)]^{n-r} \left\{ \frac{-PQ}{(P-Q)^2} + \frac{P+Q}{P-Q} f(x) \right\} dx$$

$$= -\frac{(n-1)PQ}{(r-1)(P-Q)^2} \left[\mu_{r-1:n-2}^{(k)} - \mu_{r-2:n-2}^{(k)} \right] + \frac{k}{(r-1)(P-Q)} \mu_{r-1:n-1}^{(k)}.$$

That is,

$$\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} - \frac{(n-1)PQ}{(r-1)(P-Q)^2} \left[\mu_{r-1:n-2}^{(k)} - \mu_{r-2:n-2}^{(k)} \right]$$

$$+ \frac{k}{(r-1)(P-Q)} \mu_{r-1:n-1}^{(k-1)} \tag{3.29}$$

If we set $P = 1 \, and \; Q = 0$ in (3.29), we get

$$\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + \frac{k}{r-1} \mu_{r-1:n-1}^{(k-1)}. \tag{3.30}$$

This expression for non-truncated logistic distribution was obtained by Shah (1970).

3.6. Doubly Truncated Log-logistic Distribution: (Al-Shboul & Khan, 1989)

$$f(x) = \frac{p\theta x^{p-1}}{(P-Q)(1+\theta x^p)^2}, \; Q_1 \le x \le P_1 \tag{3.31}$$

where $\int_{-\infty}^{Q_1} f(x)dx = Q, and \int_{P_1}^{\infty} f(x)dx = 1-P.$

Therefore,

$$\theta Q_1^p = \frac{Q}{(1-Q)}, \quad \theta P_1^p = \frac{P}{(1-P)}$$

and

$$Q = \frac{\theta Q_1^p}{1+\theta Q_1^p}, \quad P = \frac{\theta P_1^p}{1+\theta P_1^p}. \tag{3.32}$$

Thus

$$F(x) = \left[\frac{(1-Q)}{(P-Q)} - \frac{1}{(P-Q)(1+\theta x^p)}\right]$$

and

$$1-F(x) = \left[\frac{1}{(P-Q)(1+\theta x^p)} - \frac{(1-P)}{(P-Q)}\right]$$

It can easily be proved that

$$F(x)[(1-F(x)] = -\frac{P(1-P)}{(P-Q)^2} + \frac{(1-Q-P)}{(P-Q)}[1-F(x)] + \frac{x}{p(P-Q)}f(x) \tag{3.33}$$

$$[1-F(x)]^2 = \frac{P(1-P)}{(P-Q)^2} + \frac{(2P-1)}{(P-Q)}[1-F(x)] - \frac{x}{p(P-Q)}f(x) \tag{3.34}$$

Al-Shboul & Khan (1989) obtained

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = -\frac{Q(1-Q)(n-1)}{(P-Q)^2(r-1)}\left[\mu_{r-1:n-2}^{(k)} - \mu_{r-2:n-2}^{(k)}\right]$$

$$+\frac{(1-Q-P)(n-1)}{(P-Q)(r-1)}\left[\mu_{r-1:n-1}^{(k)}-\mu_{r-1:n-2}^{(k)}\right]$$

$$+\frac{k}{P(P-Q)(r-1)}\mu_{r-1:n-1}^{(k)} \qquad (3.35)$$

By rearranging the terms, they got

$$\mu_{r:n}^{(k)}=\left[1+\frac{k}{p(P-Q)(r-1)}+\frac{(1-Q-P)(n-1)}{(P-Q)(r-1)}\right]\mu_{r-1:n-1}^{(k)}$$

$$-\left[\frac{P(1-P)(n-1)}{(P-Q)^2(r-1)}\right]\mu_{r-1:n-2}^{(k)}+\left[\frac{Q(1-Q)(n-1)}{(P-Q)^2(r-1)}\right]\mu_{r-2:n-2}^{(k)}$$

$$2\le r\le n. \qquad (3.36)$$

For obtaining this, Al-Shboul & Khan (1989) have used the convention $\mu_{n-1:n-2}^{(k)}=P_1^k$, $\mu_{0:n-2}^{(k)}=Q_1^k$, $n\ge 2$

For $n\ge 2$

$$\mu_{1:n}^{(k)}=\left[\frac{(2P-1)}{(P-Q)}-\frac{k}{p(P-Q)(n-1)}\right]\mu_{1:n-1}^{(k)}+\left[\frac{P(1-P)}{(P-Q)^2}\right]\mu_{1:n-2}^{(k)}$$

$$-\left[\frac{Q(1-Q)}{(P-Q)^2}\right]Q_1^k \qquad (3.37)$$

Also for $k<p,\ p>0$

$$\mu_{1:1}^{(k)}=\frac{\theta^{-\frac{k}{p}}}{(P-Q)}\left[B_{1-Q}\left(1-\frac{k}{p},1+\frac{k}{p}\right)-B_{1-P}\left(1-\frac{k}{p},1+\frac{k}{p}\right)\right] \qquad (3.38)$$

where $B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt,\ a,b>0$

At $Q=0, P=1$, these results reduce to the results obtained by

Ali & Khan (1987). A corresponding result for $\mu_{r:n}^{(k-p)}$ in terms of

$\mu_{r:n}^{(k)}$ can also be obtained from Khan & Khan (1987).

3.7. Gamma Distributions: (Khan *et al.*, 1983a)

$$f(x)=\frac{e^{-x}x^{p-1}}{\Gamma(p)}, \qquad x, p>0 \tag{3.39}$$

$$\begin{aligned} 1-F(x) &= \sum_{j=0}^{p-1}\frac{e^{-x}x^{j}}{j!} \\ &= f(x)p!\sum_{j=0}^{p-1}\frac{x^{j+1-p}}{j!}. \end{aligned} \tag{3.40}$$

Thus,

$$\mu_{1:n}^{(k)}=\left(\frac{k}{n}\right)p!\sum_{j=0}^{p-1}\frac{\mu_{1:n}^{(j+k-p}}{j!}, \tag{3.41}$$

$$\mu_{r:n}^{(k)}=\mu_{r-1:n-1}^{(k)}+\left(\frac{k}{n}\right)p!\sum\frac{\mu_{r:n}^{(j+k-p)}}{j!}. \tag{3.42}$$

These expressions were obtained by Joshi (1979b). It may be noted that for $j+k<p$, some negative moments will also be involved. Joshi (1979b) has tabulated the negative the negative moments of these order statistics. Results of related interest may also be found in Gupta (1960) , Krishnaiah & Rizvi (1967) and Khan *et al.* (1984). For $p=1$, (3.42) gives the recurrence relation for a non-truncated exponential distribution.

3.8. Beta Distribution: (Khan *et al.*, 1983a)

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}; \quad a,b>0 \quad 0<x<1. \tag{3.43}$$

$$1-F(x) = \frac{1}{B(a,b)} \int_x^1 x^{a-1}(1-x)^{b-1}dx = \sum_{j=0}^{a-1}\binom{a+b-1}{j} x^j (1-x)^{a+b-1-j}.$$

$$= f(x)B(a,b)\sum_{j=0}^{a-1}\binom{a+b-1}{j} x^{j+1-a}(1-x)^{a-j}$$

$$= f(x)B(a,b)\sum_{j=0}^{a-1}\sum_{i=0}^{a-j}(-1)^i\binom{a+b-1}{j}\binom{a-j}{i} x^{j+i+1-a} \tag{3.44}$$

Thus,

$$\mu_{r:n}^{(k)} - \mu_{r-k:n-1}^{(k)} = \frac{k}{n} B(a,b)\sum_{j=0}^{a-1}\sum_{i=0}^{a-j}(-1)^i\binom{a+b-1}{j}\binom{a-j}{i}\mu_{r:n}^{(k+j+i-a)}$$

This result is valid even for $r=1$ with $\mu_{0:n}^{(k)}=0$.

3.10. Doubly Truncated Burr Distribution: (Khan & Khan, 1987)

$$f(x) = \frac{mp\theta x^{p-1}(1+\theta x^p)^{-(m+1)}}{P-Q}, \quad Q_1 \le x \le P_1 \tag{3.45}$$

where Q and $(1-P)$, $(Q<P)$ are the proportions of truncation on the left and the right of the distribution respectively and

$$\theta Q_1^p = [(1-Q)^{-1/m} - 1]$$
$$\theta P_1^p = [(1-P)^{-1/m} - 1]$$

$$1-F(x) = -\frac{1-P}{P-Q} + \frac{x^{1-p}}{mp\theta} f(x) + \frac{x}{mp} f(x) \tag{3.46}$$

Thus for $2 \le r \le n-1$ and $k \ne mnp$,

$$\left(1-\frac{k}{mnp}\right)\mu_{r:n}^{(k)}=\frac{1-Q}{P-Q}\mu_{r-1:n-1}^{(k)}-\frac{1-P}{P-Q}\mu_{r:n-1}^{(k)}+\frac{k}{mnp\theta}\mu_{r:n}^{(k-p)} \quad (3.47)$$

The important deductions for $k \neq mnp$, in view of (2.7) are:

$$\left(1-\frac{k}{mp}\right)\mu_{1:1}^{(k)}=\frac{1-Q}{P-Q}Q_1^k-\frac{1-P}{P-Q}P_1^k+\frac{k}{mp\theta}\mu_{1:1}^{(k-p)} \quad (3.48)$$

$$\left(1-\frac{k}{mnp}\right)\mu_{1:n}^{(k)}=\frac{1-Q}{P-Q}Q_1^k-\frac{1-P}{P-Q}\mu_{1:n-1}^{(k)}+\frac{k}{mnp\theta}\mu_{1:n}^{(k-p)},\ n>1 \quad (3.49)$$

$$\left(1-\frac{k}{mnp}\right)\mu_{n:n}^{(k)}$$

$$=\frac{1-Q}{P-Q}\mu_{n-1:n-1}^{(k)}-\frac{1-P}{P-Q}P_1^k+\frac{k}{mnp\theta}\mu_{n:n}^{(k-p)},\ n>1 \quad (3.50)$$

At $k=mnp$, from (3.47), we have

$$\mu_{r:n-1}^{(k)}=\frac{1-Q}{1-P}\mu_{r-1:n-1}^{(k)}+\frac{P-Q}{\theta(1-P)}\mu_{r:n}^{(k-p)},\quad 2\leq r\leq n-1 \quad (3.51)$$

$\mu_{1:1}^{(k)}=E(X^k)$ can be seen to be equal to

$$\frac{m}{(P-Q)\theta^m}\int_{(1+\theta P_1^p)^{-1}}^{(1+\theta Q_1^p)^{-1}} y^{-1}(1-y)^m\,dy \quad (3.52)$$

at $k=mp$. This can be expressed in terms of incomplete beta function or binomial sums.

Remark. The results obtained here are also true for

Lomax : $\left\{p=1,\theta=\frac{1}{a},F(x)=1-\left(1+\frac{x}{a}\right)^{-m}\right\},$

Weibull-Gamma : $\left\{\theta=\frac{1}{\delta}, F(x)=1-\left(1+\frac{x^p}{\delta}\right)^{-m}\right\}$,

Weibull-Exponential : $\left\{m=1, \theta=\frac{1}{\delta}, F(x)=\frac{\frac{x^p}{\delta}}{(1+\frac{x^p}{\delta})}\right\}$ and

Log logistic : $\left\{m=1, \theta=a^{-p}, F(x)=\frac{\left(\frac{x}{a}\right)^p}{1+\left(\frac{x}{a}\right)^p}\right\}$

distributions.

3.9. Doubly Truncated Generalized Exponential Distribution: (Saran & Pushkarna, 1999)

The *pdf* of generalized exponential distribution is given by

$$g(x)=(1-\alpha x)^{(1/\alpha)-1}, \quad 0\le x\le\frac{1}{\alpha}, \; 0\le\alpha\le1 \tag{3.53}$$

and *cdf* is given by

$$G(x)=1-(1-\alpha x)^{1/\alpha}, \quad 0\le x\le\frac{1}{\alpha}, \; 0\le\alpha\le1 \tag{3.54}$$

where α is the shape parameter.

Truncated *pdf* is given by,

$$f(x)=\frac{(1-\alpha x)^{(1/\alpha)-1}}{P-Q}, \quad Q_1\le x\ge P_1, \; 0\le\alpha\le1 \tag{3.55}$$

and *cdf* is given by

$$F(x)=\frac{(1-\alpha Q_1)^{1/\alpha}-(1-\alpha x)^{1/\alpha}}{P-Q}, \quad Q_1 \le x \le P_1 \tag{3.56}$$

Here $Q_1=\dfrac{1-(1-Q)^{\alpha}}{\alpha}$ and $P_1=\dfrac{1-(1-P)^{\alpha}}{\alpha}$ (3.57)

Saran & Pushkarna (1999) obtained the recurrence relation given by

$$\mu_{r:n}^{(k)}=\frac{1}{n+\alpha k}(nQ_2\mu_{r-1:n-1}^{(k)}-nP_2\mu_{r:n-1}^{(k)}+k\mu_{r:n}^{(k-1)})$$

$$2 \le r \le n-1 \tag{3.58}$$

From (3.53), we have

$$\mu_{1:n}^{(k)}=\frac{1}{n+\alpha k}(nQ_2Q_1^k-nP_2\mu_{1:n-1}^{(k)}+k\mu_{1:n}^{(k-1)}), \quad n \ge 2 \tag{3.59}$$

$$\mu_{n:n}^{(k)}=\frac{1}{n+\alpha k}(nQ_2\mu_{n-1:n-1}^{(k)}-nP_2P_1^k+k\mu_{n:n}^{(k-1)}), \quad n \ge 2 \tag{3.60}$$

and

$$\mu_{1:1}^{(k)}=E(X^k)=\frac{1}{1+\alpha k}(Q_2Q_1^k-nP_2P_1^k+k\mu_{1:1}^{(k-1)}) \tag{3.61}$$

If the shape parameter $\alpha \to 0$, then the pdf in (3.53) becomes exponential distribution.

4. Recurrence Relation for General form of Distributions for single Moments of order statistics:

Khan & Ali (1997) have used the results of Theorem 2.1 and Theorem 2.2 to obtain the relations and identities for some general form of distributions.

4.1. $\mathbb{F}_1(\mathbf{x}) = \mathbf{1} - [\mathbf{ah}(\mathbf{x}) + \mathbf{b}]^{\mathbf{c}}; \quad x \in (\alpha, \beta)$ (4.1)

where $a \neq 0, b, c \neq 0$ are finite constants and $h(x)$ is continuous, monotonic and differentiable function of x.

Then the truncated *pdf* $f(x)$ is given by

$$f(x) = -\frac{ca}{P-Q}[ah(x)+b]^{c-1}h'(x); \qquad x \in (Q_1, P_1) \tag{4.2}$$

and the corresponding truncated *df* $F(x)$ by

$$1 - F(x) = -P_2 - \frac{ah(x)+b}{cah'(x)}f(x), \qquad P_2 = \frac{1-P}{P-Q} \tag{4.3}$$

$E\{g(X_{r:n})\}$ satisfy the following relations and identities for the *df* $F(x)$ in (4.3).

Theorem4.1 (Khan & Ali, 1997) For $1 \le r \le n; n = 1,2\ldots$

$$E\{g(X_{r:n})\} = (1+P_2)E\{g(X_{r-1:n-1})\} - P_2E\{g(X_{r:n-1})\} - \frac{1}{nca}E\{m(X_{r:n})\}$$

$$where\ m(x) = [ah(x)+b]W(x),\ W(x) = \frac{g'(x)}{h'(x)} \tag{4.4}$$

Proof: From Theorem 2.1 and (4.3) , we have

$$E\{g(X_{r:n})\} - E\{g(X_{r-1:n-1})\}$$

$$= -\binom{n-1}{r-1}\int_{Q_1}^{P_1} g'(x)[F(x)]^{r-1}[1-F(x)]^{n-r}\left[P_2 + \frac{ah(x)+b}{cah'(x)}f(x)\right]dx \tag{4.5}$$

and hence the result.

Theorem 4.2(Khan & Ali, 1997) For $1 \le r \le n; \quad n = 1,2,\ldots$

$$E\{g(X_{r:n})\} = E\{g(X_{r-1:n-1})\} - \frac{(P-Q)(n-r+1)}{n(n+1)ca}E\{Z(X_{r:n+1})\} \tag{4.6}$$

where $Z(x)=[ah(x)+b]^{1-c}W(x), \quad W(x)=\frac{g'(x)}{h'(x)}$.

Proof : From (4.2), we have

$$1=-\frac{(P-Q)[ah(x)+b]^{1-c}}{cah'(x)}f(x) \tag{4.7}$$

Now from Theorem 2.1 and (4.7), we have

$$E\{g(X_{r:n})\}-E\{g(X_{r-1:n-1})\}$$

$$=-\frac{(P-Q)}{ca}\binom{n-1}{r-1}\int_{Q_1}^{P_1}Z(x)[F(x)]^{r-1}[1-F(X)]^{n-r+1}f(x)dx \tag{4.8}$$

and hence the result.

Similarly, we have

$$E\{g(X_{r:n})\}=E\{g(X_{r-1:n})\}-\frac{(P-Q)}{(n+1)ca}E\{Z(Xr:n+1)\} \tag{4.9}$$

Recurrence relations for moments of order statistics for the Pareto, power function, Weibull, Burr type XII, beta of first and Cauchy distributions may be obtained by proper choice of a,b,c and $h(x)$ as given by Khan and Abu-Salih(1989).

4.2. $\mathbf{F_1(x)=[ah(x)+b]^c}$, $x\in(\alpha,\beta)$ (4.10)

where $a\neq 0, b, c\neq 0$ are finite constants and $h(x)$ is continuous, monotonic and differentiable function of x.

Then the truncated *pdf* $f(x)$ is given by

$$f(x)=\frac{ca}{P-Q}[ah(x)+b]^{c-1}h'(x); \qquad x\in(Q_1,P_1) \tag{4.11}$$

and the corresponding truncated *df* $F(x)$ by

$$1-F(x)=P_3-\frac{ah(x)+b}{cah'(x)}f(x), \qquad P_3=\frac{P}{P-Q} \tag{4.12}$$

Now $E\{g(X_{r:n})\}$ satisfy the following relations and identities for the *df* $F(x)$ given in (4.12).

Theorem 4.3(Khan & Ali, 1997) For $1\le r\le n; n=1,2,\ldots$

$$E[\{g(X_{r:n}\}]=(1-P_3)E\{g(X_{r-1:n-1})\}$$

$$+P_3E\{g(X_{r:n-1})\}-\frac{1}{nca}E\{m(X_{r:n})\} \tag{4.13}$$

where $m(x)=[ah(x)+b]W(x)$, $W(x)=\dfrac{g'(x)}{h'(x)}$.

Proof : The proof is simple.

Theorem 4.4 (Khan & Ali, 1997) For $1\le r\le n; \quad n=1,2,\ldots$

$$E\{g(X_{r:n})\}=E\{g(X_{r-1:n})\}-\frac{(P-Q)}{(n+1)ca}E\{Z(X_{r:n+1})\} \tag{4.14}$$

where $Z(x)=[ah(x)+b]^{1-c}W(x), \quad W(x)=\dfrac{g'(x)}{h'(x)}$.

Proof : From (4.11), we have

$$1=\frac{(P-Q)[ah(x)+b]^{1-c}}{cah'(x)}f(x) \tag{4.15}$$

and the proof is obvious.

Recurrence relations for moments of order statistics for the Pareto, power function, inverse Weibull, Burr type III, Cauchy distributions may be obtained by proper choice of a, b, c and $h(x)$ as given by Khan & Abu-Salih(1989)

4.3. $F_1(x)=1-be^{-ah(x)}$, $x \in (\alpha, \beta)$ (4.16)

where $a \neq 0, b > 0$ are constants and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha, \beta]$.Then the truncated *pdf* $f(x)$ is given by

$$f(x)=\frac{ab}{P-Q}e^{-ah(x)}h'(x); \qquad x \in (Q_1, P_1) \tag{4.17}$$

and the corresponding truncated *df* $F(x)$ by

$$1-F(x)=-P_2+\frac{1}{ah'(x)}f(x), \qquad P_2=\frac{1-P}{P-Q} \tag{4.18}$$

Now $E\{g(X_{r:n})\}$ satisfy the following relations and identities for the *df* $F(x)$ given in (4.18).

Theorem 4.5(Khan & Ali, 1997) For $1 \leq r \leq n; n=1,2,\ldots$

$$E[\{g(X_{r:n}\}]=(1+P_2)E\{g(X_{r-1:n-1})\}$$

$$-P_2E\{g(X_{r:n-1})\}+\frac{1}{na}E\{W(X_{r:n})\} \tag{4.19}$$

Proof: From Theorem 2.1 and (4.18), we have

$$E\{g(X_{r:n})\}-E\{g(X_{r-1:n-1})\}$$

$$=-P_2\binom{n-1}{r-1}\int_{Q_1}^{P_1} g'(x)[F(x)]^{r-1}[1-F(x)]^{n-r}\,dx$$

$$+\frac{1}{a}\binom{n-1}{r-1}\int W(x)[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

Now on using Theorem 2.1 and (2.12) we can get the required result.

Theorem 4.6(Khan & Ali, 1997) For $1 \le r \le n;\ n = 1,2,\ldots$

$$\left(1 - \frac{r-1}{n-r+1} P_2\right) E\{g(X_{r:n})\} = (1 + P_2) E\{g(X_{r-1:n})\}$$
$$- \frac{nP_2}{n-r+1} E\{g(X_{r:n-1})\} + \frac{1}{(n-r+1)a} E\{W(X_{r:n})\} \tag{4.20}$$

Proof: Starting from Theorem 2.2 and (4.18), we can prove it easily.

Recurrence relations for moments of order statistics for the Pareto, Weibull, Burr type XII and beta of first kind distributions may be obtained from these results by adjusting a, b and $h(x)$ as given by Khan and Abu-Salih(1989) with b replaced by e^{b}.

4.4. $\mathbf{F_1(x) = be^{-h(x)}}$; $x \in (\alpha, \beta)$ (4.21)

where $a \ne 0, b > 0$ are constants and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha, \beta]$.

Then the truncated *pdf* $f(x)$ is given by

$$f(x) = -\frac{a^{bq}}{P-Q} e^{-ah(x)} h'(x); \qquad x \in (Q_1, P_1) \tag{4.22}$$

and the corresponding truncated *df* $F(x)$ by

$$1 - F(x) = P_3 + \frac{1}{ah'(x)} f(x), \qquad P_3 = \frac{P}{P-Q} \tag{4.23}$$

The single moments $E\{g(X_{r:n})\}$ satisfy the following relations and identities for the *df* $F(x)$ given in (4.23).

Theorem 4.7 (Khan & Ali, 1997) For $1 \le r \le n; n = 1,2,\ldots\ldots$

$$E[\{g(X_{r:n}\}] = (1 - P_3) E\{g(X_{r-1:n-1})\}$$
$$+ P_3 E\{g(X_{r:n-1})\} + \frac{1}{na} E\{W(X_{r:n})\} \tag{4.24}$$

Proof: The proof is easy.

Theorem 4.8 (Khan & Ali, 1997) For $1 \le r \le n;\ n = 1,2,.....$

$$\left(1+\frac{r-1}{n-r+1}P_3\right)E\{g(X_{r:n})\}=(1-P_3)E\{g(X_{r-1:n})\}$$
$$+\frac{nP_3}{n-r+1}E\{g(X_{r:n-1})\}+\frac{1}{(n-r+1)a}E\{W(X_{r:n})\} \tag{4.25}$$

Proof: The proof is straightforward..

Recurrence relations for moments of order statistics for the Pareto, power function, inverse Weibull and Burr type III distributions may be obtained by proper choice of a,b and $h(x)$ with b replaced by e^b as given by Khan & Abu-Salih(1989).

Remark1: The recurrence relations between the moments, moment generating functions, characteristic functions, and distribution functions (truncated and non-truncated),whenever they exist can be obtained by setting $g(x)$ equal to $x^k, e^{tx}, e^{itx}, and\, I_{[Q_1,x]}(X)$ respectively.

Remark2: Particular cases of interest for which relations may be obtained are Rectangular distribution (from beta of the first kind $1-(1-x)^p\ at\ p=1$), Rayleigh and exponential distributions (from Weibull $1-\exp(-\theta x^p)\ at\ p=2\ and\ p=1$ respectively), Lomax and log-logistic distributions (from Burr type XII $1-(1+\theta x^p)^{-m}\ at\ p=1\ and\ m=1$ respectively).

5. A General Recurrence Relation for Moments of Order Statistics in a Class Of Probability Distribution:

Consider the class $\Im$ of distribution functions F given by

$$(F^{-1})'(t)=\frac{1}{d}t^p(1-t)^{q-p-1} \qquad a.e.\ on\ (0,1) \tag{5.1}$$

with a constant $d > 0$, and integers p and q.

5.1 The Recurrence Relations: (Kamps, 1991)

Theorem 5.1 (Kamps, 1991) Let X be a random variable with distribution function $F \in \Im$, $m \geq 1$ a constant, and $F^{-1}(0+) \geq 0$, if $m \notin \mathrm{N}$. Then for all $k,\ n \in \mathrm{N}, 2 \leq k \leq n$, satisfying

$1 \leq k + p \leq n + q,$

and

$$-\infty < E(X_{k:n}^{m}), E(X_{k-1:n}^{m}), E(X_{k+p:n+q}^{m-1}) < \infty,$$

the identity

$$E(X_{k:n}^{m}) - E(X_{k-1:n}^{m}) = mc(k,n,p,q)E(X_{k+p:n+q}^{m-1})$$

with the constants $c(k,n,p,q)$ given by

$$c(k,n,p,q) = E(X_{k:n}) - E(X_{k-1:n}) = \frac{1}{d}\frac{\binom{n}{k-1}}{(k+p)\binom{n+q}{k+p}}$$

holds true.

Proof. Since we have

$$E(X_{k:n}^{m}) = k\binom{n}{k}\int_{0}^{1}\{F^{-1}(t)\}^{m} t^{k-1}(1-t)^{n-k}\,dt,$$

integration by parts yields

$$E(X_{k:n}^{m}) - E(X_{k-1:n}^{m})$$

$$= m\binom{n}{k-1}\int_0^1 \{F^{-1}(t)\}^{m-1}(F^{-1})'(t)t^{k-1}(1-t)^{n-k+1}dt$$

(*cf.* Khan *et al.*, 1983a, Lin 1988).

Putting in the representations of $(F^{-1})'(t)$ and $c(k,n,p,q)$ the assertion follows.

Furthermore, the equality

$$E(X_{k:n}) - E(X_{k-1:n}) = c(k,n,p,q)$$

holds true, because of using the above formula for expectation of the spacing $X_{k:n} - X_{k-1:n}$(David, Groeneveld 1982), we find

$$E(X_{k:n}) - E(X_{k-1:n}) = \binom{n}{k-1}\int_0^1 (F^{-1})'(t)t^{k-1}(1-t)^{n-k+1}dt$$

$$= \frac{1}{d}\binom{n}{k-1}\int_0^1 t^{k+p-1}(1-t)^{n-k+q-p}dt$$

$$= \frac{1}{d}\frac{\binom{n}{k-1}}{(k+p)\binom{n+q}{k+p}} = c(k,n,p,q).$$

A modification of this recurrence relation, and by this a modification of the following theorem too, is obtained observing that the equation

$$E(X_{k:n}^m) - E(X_{k-1:n}^m) = \frac{n}{n-k+1}E(X_{k:n}^m) - E(X_{k-1:n-1}^m),\ 2 \le k \le n,$$

is valid for an arbitrary distribution.

It is deduced from the well-known relationship

$$E(X_{k-1:n-1}^{m}) = \frac{n-k+1}{n} E(X_{k-1:n}^{m}) - \frac{k-1}{n} E(X_{k:n}^{m}),\ 2 \le k \le n,$$

for order statistics of a random sample of size n from an arbitrary distribution (Cole 1951, David 1981).

Noticing this modification, the representation of $E(X_{k:n}^{m}) - E(X_{k-1:n}^{m})$ used in the proof has been stated in the Theorem 2 of Khan *et al.* (1983a).

Recurrence relations for various distributions can be obtained by adjusting p, q as given in the following table.

p	q	$F(x)$	$x \in$	Distribution	$d.c(k,n,p,q)$
0	0	$1-\exp\{-d(x-c)\}$	(c,∞)	exponential	$\frac{1}{n-k+1}$
0	>0	$1-\{dq(c-x)\}^{1/q}$	$\left(c-\frac{1}{dq},c\right)$		$\frac{n!(n-k+q)!}{(n+q)!(n-k+1)!}$
0	<0	$1-\{dq(c-x)\}^{1/q}$	$\left(c-\frac{1}{dq},\infty\right)$	Pareto, Lomax	$\frac{n!(n-k+q)!}{(n+q)!(n-k+1)!}$
-1	0	$\exp\{d(x-c)\}$	$(-\infty,c)$		$\frac{1}{k-1}$
>-1	$p+1$	$\{dq(x-c)\}^{1/q}$	$\left(c,c+\frac{1}{dq}\right)$	power	$\frac{n!(k+p-1)!}{(n+p+1)!(k-1)!}$
<-1	$p+1$	$\{dq(x-c)\}^{1/q}$	$\left(-\infty,c+\frac{1}{dq}\right)$		$\frac{n!(k+p-1)!}{(n+p+1)!(k-1)!}$
-1	-1	$[1+\exp\{-d(x-c)\}]^{-1}$	$(-\infty,\infty)$	logistic	$\frac{n}{(k-1)(n-k+1)}$

Chapter 3

Recurrence Relations between Product Moments of Order Statistics

1. Introduction:

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a population with distribution function $F(x)$. Let $X_{r:n}$ and $X_{s:n}$ be the rth and the sth order statistics, then the product moment may be as

$$\mu_{r,s:n}^{(j,k)} = E(X_{r:n}^{j} X_{s:n}^{k})$$

Thus covariance is defined as

$$\sigma_{r,s:n} = E(X_{r:n} - \mu_{r:n})(X_{s:n} - \mu_{s:n})$$

and variance as

$$\sigma_{r:n}^{2} = E(X_{r:n} - \mu_{r:n})^{2}$$

For single moments, we have

$$\mu_{r,s:n}^{(j,0)} = E(X_{r:n}^{j}) = \mu_{r:n}^{j}$$

2. Recurrence Relations:

Theorem 2.1 Let $g(x, y)$ be a Borel measurable function of (x, y), then

for $1 \le r < s \le n,\ n = 1,2,\ldots$

$$E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\}$$

$$= \frac{C_{r,s:n}}{(n-s+1)} \iint_{Q_1 \le x < y \le P_1} \frac{\partial}{\partial y} g(x,y)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\times [1-F(y)]^{n-s+1} f(x) dx dy \qquad (2.1)$$

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

Proof. For proof see Khan & Ali (1998).

For $g(x,y) = x^j y^k$, Khan *et al.* (1983b) obtained the following results for the truncated distribution

1. For $1 \le r < s \le n, and\ j,k > 0,$

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s-1:n}^{*} k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y)-F(x)\}^{s-r-1} \{1-F(y)\}^{n-s+1} f(x) dy dx \qquad (2.2)$$

where $C_{r,s-1:n}^{*} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s+1)!} = \dfrac{C_{r,s-1:n}}{(s-r-1)}.$

2. For $1 \le r \le n-1, and\ j,k > 0,$

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r:n}^{(j+k)} + C_{r:n} k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1} \{1-F(x)\}^{n-r} f(x) dy dx$$

where $C_{r:n} = \dfrac{C_{r,n+1:n}}{(n-r)} = \dfrac{n!}{(r-1)!(n-r)!}.$ (2.3)

3. For $n > 1 and\ j,k > 0,$

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n-1:n}^{(j+k)}$$

$$+ n(n-1)k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{n-2} \{1 - F(x)\} \ f(x) dy dx \qquad (2.4)$$

Examples:

2.1. Doubly Truncated Weibull and Exponential Distributions: (Khan *et al.*, 1983b)

$$f(x) = \frac{p x^{p-1} e^{-x^p}}{P - Q}, \quad -\log(1-Q) \le x^p \le -\log(1-P), \ p > 0 \qquad (2.5)$$

Here $Q_1^p = -\log(1-Q), P_1^p = -\log(1-P)$. Let $Q_2 = (1-Q)/(1-P)$ and $P_2 = (1-P)/(P-Q)$, then

$$\begin{aligned} \{1 - F(y)\} &= -P_2 + \frac{e^{-x^p}}{P-Q} \\ &= -P_2 + \frac{1}{p} y^{1-p} f(x) \end{aligned} \qquad (2.6)$$

Putting the value of $\{1 - F(y)\}$, into (2.2), we get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s-1:n}^{*} \ k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} \left\{ -P_2 + \frac{1}{p} y^{1-p} f(y) \right\} f(x) dy dx$$

On simplification, we have

$$\mu_{r,s:n}^{(j,k)} = \mu_{r,s-1:n}^{(j,k)} - \frac{nP_2}{n-s+1}\left\{\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}\right\}$$

$$+ \frac{k}{p(n-s+1)}\mu_{r,s:n}^{(j,k-p)}, \quad 1 \le r < s \le n,\ s-r \ge 2 \tag{2.7}$$

From (2.3),for $s = r+1$ (2.7) reduces to

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r:n}^{(j+k)} - \frac{nP_2}{n-s+1}\left\{\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j+k)}\right\}$$

$$+ \frac{k}{p(n-r)}\mu_{r,r+1:n}^{(j,k-p)}, \quad 1 \le r \le n-1,\ n \ge 3 \tag{2.8}$$

From (2.4), $forr = n-1$ and $s = n,$ (2.7) reduces to

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n-1:n}^{(j+k)} - nP_2\left\{P_1^k\mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)}\right\}$$

$$+ \frac{k}{p}\mu_{n-1,n:n}^{(j,k-p)}, \quad n \ge 2. \tag{2.9}$$

Expression (2.9) could also have been obtained from (2.8) by putting $r = n-1$.In substitution authors got a term $\mu_{n-1,n:n-1}^{(j,k)}$ which is essentially an undefined term. This can be interpreted as $E\left(X_{n-1:n-1}^{j}P_1^k\right) = P_1^k \mu_{n-1:n-1}^{(j)}$, where P_1 is the upper limit of the Weibull variate. However, they reached at this conclusion after doing actual calculations as given bellow:

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n-1:n}^{(j+k)} - nP_2(n-1)k\int_{Q_1}^{P_1} x^j\{F(x)\}^{n-2}\left(\int_x^{P_1} y^{k-1}dy\right)f(x)dx$$

$$+\frac{n(n-1)k}{p}\int_{Q_1}^{P_1}\int_x^{P_1} x^j y^{k-p}\{F(x)\}^{n-2} f(x)f(y)dydx$$

$$= \mu_{n-1:n}^{(j+k)} - nP_2(n-1)\int_{Q_1}^{P_1}(P_1^k - x^k)x^j\{F(x)\}^{n-2} f(x)dx$$

$$+\frac{k}{p}\mu_{n-1,n:n}^{(j,k-p)}$$

$$= \mu_{n-1:n}^{(j+k)} - nP_2\left\{P_1^k\mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)}\right\}+\frac{k}{p}\mu_{n-1,n:n}^{(j,k-p)}$$

By putting $p = 1$ in the above expressions, the authors obtained the corresponding results for exponential distribution. For non-truncated case one has to put $P = 1$ *and* $Q = 0$. In case of the doubly truncated Weibull distribution, recurrence relations for $\mu_{r:n}^{(k)}$ are available in

Khan *et al.* (1983a). Expressions for exact and explicit product moment with $j = k = 1$ can be obtained in Lieblein (1955).

In case of $j = k = 1$, for the Weibull distribution, Khan *et al.* (1983b)

obtained the following expression.

$$\mu_{r,s:n} = \mu_{r,s-1:n} - \frac{nP_2}{n-s+1}\{\mu_{r,s:n-1} - \mu_{r,s-1:n-1}\}$$

$$+\frac{1}{p(n-s+1)}\mu_{r,s:n}^{(1,1-p)}, \quad 1 \le r < s \le n,\ s-r \ge 2 \tag{2.10}$$

For the exponential distribution, this reduces to

$$\mu_{r,s:n} = \mu_{r,s-1:n} - \frac{nP_2}{n-s+1}\{\mu_{r,s:n-1} - \mu_{r,s-1:n-1}\}$$

$$+ \mu_{r:n/(n-s+1)}, \quad 1 \le r < s \le n, \; s-r \ge 2 \tag{2.11}$$

From (2.7), it is clear that if $k < p$, then the power of Y will be negative.

2.2. Doubly Truncated Power Function Distribution: (Khan *et al.*, 1983b)

$$f(x) = \frac{va^{-v}x^{v-1}}{P-Q}, \quad aQ^{1/v} \le x \le aP^{1/v}, \; a, v > 0 \tag{2.12}$$

Here, $Q_1 = aQ^{1/v}$, $P_1 = aP^{1/v}$.

Let $P_2 = P/(P-Q)$ and $Q_2 = Q/(P-Q)$. Then

$$\{1 - F(y)\} = P_2 - \frac{y}{v} f(y).. \tag{2.13}$$

Putting the value of $\{1 - F(y)\}$, into (2.2), Khan *et al.* (1983b) deduced

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s-1:n}^{*} \; k \int_{Q_1}^{P_1}\int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} \left\{P_2 - \frac{y}{v} f(y)\right\} f(x) dy dx$$

On simplification, Khan *et al.* (1983b) got

$$\mu_{r,s:n}^{(j,k)} = \frac{v}{v(n-s+1)+k}\left[(n-s+1)\mu_{r,s-1:n}^{(j,k)} + nP_2(\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)})\right]$$

$$1 \le r < s \le n, \; s-r \ge 2 \tag{2.14}$$

For $s = r+1$, they got from Corollary 1,

$$\mu_{r,r+1:n}^{(j,k)} = \frac{v}{v(n-r)+k}\left[(n-r)\mu_{r:n}^{(j+k)} + nP_2(\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j+k)})\right]$$

$$1 \le r \le n-2,\ n \ge 3 \qquad (2.15)$$

after noting that $\mu_{r,r:n}^{(j,k)} = \mu_{r:n}^{(j+k)}$.

Similarly for $n = s = r+1$, they got

$$\mu_{n-1,n:n}^{(j,k)} = \frac{v}{v+k}\left[\mu_{n-1:n}^{(j+k)} + nP_2(P_1^k \mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)})\right] n \ge 2, \qquad (2.16)$$

interpreting $\mu_{n-1,n:n-1}^{(j,k)} = P_1^k \mu_{n-1:n-1}^{(j)}$ as discussed in example 5.1.

For $j = k = 1$, the relations are available in Balakrishnan & Joshi (1981b). The non-truncated cases are discussed by Malik (1967). Reference may also be made to Khan *et al.* (1983a) for the recurrence relation of $\mu_{r:n}^{(i)}$, $i = 1,2,\ldots$.

2.3. Doubly Truncated Pareto Distribution: (Khan *et al.*, 1983b)

$$f(x) = \frac{va^v x^{-v-1}}{P-Q}, \quad a(1-Q)^{-1/v} \le x \le a(1-P)^{-1/v},\ a, v > 0 \qquad (2.17)$$

Here, $Q_1 = a(1-Q)^{-1/v}$, $P_1 = a(1-P)^{-1/v}$. Let

$P_2 = (P-1)/(P-Q)$ *and* $Q_2 = (Q-1)/(P-Q)$. Then

$$\{1 - F(y)\} = \frac{y}{v} f(y) + P_2. \qquad (2.18)$$

In view of (2.18) and (2.2), they get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s-1:n}^{*}\ k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} \left\{ \frac{y}{v} f(y) + P_2 \right\} f(x) dy dx$$

On simplification, Khan *et al.* (1983b) got

$$\mu_{r,s:n}^{(j,k)} = \frac{v}{v(n-s+1)-k} \left[(n-s+1)\mu_{r,s-1:n}^{(j,k)} + nP_2(\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}) \right],$$

$1 \le r < s \le n,\ s - r \ge 2$ and $v(n-s+1) \ne k$ (2.19)

However, if $k = v(n-s+1)$, they got from (2.23)

$$(n-s+1)\mu_{r,s-1:n}^{(j,k)} = nP_2(\mu_{r,s-1:n-1}^{(j,k)} - \mu_{r,s:n-1}^{(j,k)} \qquad (2.20)$$

Marginal results for $k \ne v(n-s+1)$ can easily be seen to be equal to

$$\mu_{r,r+1:n}^{(j,k)} = \frac{v}{v(n-r)-k} \left[(n-r)\mu_{r:n}^{(j+k)} + nP_2(\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j+k)}) \right],$$

$1 \le r \le n-2,\ n \ge 3$ (2.21)

$$\mu_{n-1,n:n}^{(j,k)} = \frac{v}{v-k} \left[\mu_{n-1:n}^{(j+k)} + nP_2(P_1^k \mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)}) \right] n \ge 2, \qquad (2.22)$$

Recurrence relation for $j = k = 1$ have been studied by Balakrishnan & Joshi (1982). Malik (1966) has obtained these results for $P = 1, Q = O$. To evaluate $\mu_{r,s:n}^{(j,k)}$, one may require the recurrence relations for $\mu_{r:n}^{(i)}$ for which one may refer to Khan *et al.* (1983a).

2.4. Doubly Truncated Cauchy Distribution: (Khan *et al.*, 1983b)

$$f(x)=\frac{1}{(P-Q)\pi}\frac{1}{1+x^2}, \qquad Q_1 \le x \le P_1, \tag{2.23}$$

where Q_1 and P_1 are obtained by

$$\int_{-\infty}^{Q_1} f(x)dx = Q, and \int_{P_1}^{\infty} f(x)dx = 1-P.$$

Therefore,

$$(P-Q)\pi(1+x^2)f(x)=1 \tag{2.24}$$

In view of (2.2) and (2.24),

$$\mu_{r,s:n}^{(j,k)}-\mu_{r,s-1:n}^{(j,k)}=C_{r,s-1:n}^{*}\ k\pi(P-Q)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^j y^{k-1}(1+y^2)\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s+1} f(x)f(y)dydx$$

$$=\frac{k\pi(P-Q)}{n+1}\left[\mu_{r,s:n+1}^{(j,k-1)}+\mu_{r,s:n+1}^{(j,k+1)}\right]$$

Rearranging the terms, and replacing n by $n-1$, Khan *et al.* (1983b) got

$$\mu_{r,s:n}^{(j,k+1)}=\frac{n}{\pi k(P-Q)}\left[\mu_{r,s:n-1}^{(j,k)}-\mu_{r,s-1:n-1}^{(j,k)}\right]-\mu_{r,s:n}^{(j,k-1)}, \quad 1\le r<s\le n-1 \tag{2.25}$$

Similarly, it can be shown that

$$\mu_{r,r+1:n}^{(j,k+1)}=\frac{n}{\pi k(P-Q)}\left[\mu_{r,r+1:n-1}^{(j,k)}-\mu_{r:n-1}^{(j+k)}\right]-\mu_{r,r+1:n}^{(j,k-1)}, \quad 1\le r\le n-2 \tag{2.26}$$

and

$$\mu_{n-2,n-1:n}^{(j,k+1)} = \frac{n}{\pi k(P-Q)}\left[\mu_{n-2,n-1:n-1}^{(j,k)} - \mu_{n-2:n-1}^{(j+k)}\right] - \mu_{n-2,n-1:n}^{(j,k-1)},$$

$$n \geq 3. \qquad (2.27)$$

Putting $j = k = 1$, (2.25) reduces to

$$\mu_{r,s:n}^{(1,2)} = \frac{n}{\pi(P-Q)}\left[\mu_{r,s:n-1} - \mu_{r,s-1:n-1}\right] - \mu_{r:n}. \qquad (2.28)$$

For the non-truncated case, put $P = 1, Q = 0$. For the recurrence relation of $\mu_{r:n}^{(k)}$ in this case, one may refer to Khan *et al.* (1983a).

2.5. Doubly Truncated Burr Distribution: (Khan & Khan, 1987)

$$f(x) = \frac{m p \theta x^{p-1}(1+\theta x^{p})^{-(m+1)}}{P-Q}, \quad Q_1 \leq x \leq P_1 \qquad (2.29)$$

Khan & Khan (1987) obtained the following result.

For $1 \leq r < s \leq n-1, s-r \geq 2$ and $k \neq (n-s+1)m p$,

$$\left[1 - \frac{k}{(n-s+1)m p}\right]\mu_{r,s:n}^{(j,k)} = -\frac{n(1-P)}{(n-s+1)(P-Q)}\left[\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}\right]$$

$$+ \mu_{r,s-1:n}^{(j,k)} + \frac{k}{(n-s+1)m p \theta}\mu_{r,s:n}^{(j,k-p)} \qquad (2.30)$$

Khan & Khan (1987) deduced the following results:

$$\left[1 - \frac{k}{(n-r)m p}\right]\mu_{r,r+1:n}^{(j,k)} = -\frac{n(1-P)}{(n-r)(P-Q)}\left[\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j+k)}\right]$$

$$+\mu_{r:n}^{(j+k)}+\frac{k}{(n-r)mp\theta}\mu_{r,r+1:n}^{(j,k-p)}, k\neq(n-r)mp \text{ and } 1\leq r\leq n-2$$

$$\left[1-\frac{k}{mp}\right]\mu_{n-1,n:n}^{(j,k)}=-\frac{n(1-P)}{(P-Q)}\left[P_1^k\mu_{n-1:n-1}^{(j)}-\mu_{n-1:n-1}^{(j+k)}\right]$$

$$+\mu_{n-1:n}^{(j+k)}+\frac{k}{mp\theta}\mu_{n-1,n:n}^{(j,k-p)}, k\neq mp \text{ and } n\geq 2$$

Accordingly, we can obtain moments when $k=(n-s+1)mp$.

2.6. Doubly Truncated Generalized Exponential Distribution: (Saran & Pushkarna, 1999)

$$f(x)=\frac{(1-\alpha x)^{(1/\alpha)-1}}{P-Q}, \quad Q_1\leq x\geq P_1, \quad 0\leq\alpha\leq 1 \tag{2.31}$$

and

$$F(x)=\frac{(1-\alpha Q_1)^{1/\alpha}-(1-\alpha x)^{1/\alpha}}{P-Q}, \quad Q_1\leq x\leq P_1 \tag{2.32}$$

Here $Q_1=\frac{1-(1-Q)^{\alpha}}{\alpha}$ and $P_1=\frac{1-(1-P)^{\alpha}}{\alpha}$ (2.33)

It can be seen that

$$(1-\alpha x)f(x)=Q_2-F(x) \tag{2.34}$$

and

$$[1-F(y)]=(1-\alpha y)f(y)-P_2 \tag{2.35}$$

where,

$$Q_2=\frac{1-Q}{P-Q} \text{ and } P_2=\frac{1-P}{P-Q} \tag{2.36}$$

Putting the value of $[1-F(y)]$, into (2.2), Saran & Pushkarna (1999) obtained the following result:

For $1 \le r < s \le n-1$ and $s-r \ge 2$

$$\mu_{r,s:n}^{(j,k)} = \frac{1}{n-s+1+\alpha k}\{(n-s+1)\mu_{r,s-1:n}^{(j,k)} + k\mu_{r,s:n}^{(j,k-1)}$$

$$-nP_2(\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)})\} \tag{2.37}$$

From (2.37), we have

$$\mu_{r,r+1:n}^{(j,k)} = \frac{1}{n-r+\alpha k}\{(n-r)\mu_{r:n}^{(j+k)} + k\mu_{r,r+1:n}^{(j,k-1)}$$

$$-nP_2(\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j+k)})\}, \; 1 \le r \le n-1 \tag{2.38}$$

$$\mu_{r,n:n}^{(j,k)} = \frac{1}{1+\alpha k}\{\mu_{r,n-1:n}^{(j,k)} + k\mu_{r,n:n}^{(j,k-1)}$$

$$-nP_2(P_1^k\mu_{r:n-1}^{(j)} - \mu_{r,n-1:n-1}^{(j,k)})\}, 1 \le r \le n-1 \tag{2.39}$$

$$\mu_{n-1,n:n}^{(j,k)} = \frac{1}{1+\alpha k}\{\mu_{n-1:n}^{(j+k)} + k\mu_{n-1,n:n}^{(j,k-1)}$$

$$-nP_2(P_1^k\mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)})\}, \; n \ge 2 \tag{2.40}$$

If the shape parameter $\alpha \to 0$, then the *pdf* in (2.35) becomes exponential distribution.

3. Recurrence Relations of Product Moments for the General Form of Distributions:

Khan & Ali (1998) have obtained the recurrence relations for the product moments of order statistics for some general form of distributions.

3.1. Let us assume that the form of distribution function $F_1(x)$ is of the general form (Khan & Abu-Salih, 1989)

$$\mathbf{F_1(x) = 1 - [ah(x) + b]^c}; \quad x \in (\alpha, \beta) \tag{3.1}$$

where $a \neq 0, b, c \neq 0$ are finite constants and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha, \beta]$.

Then the truncated *pdf* $f(x)$ is given by

$$f(x) = -\frac{ca}{P-Q}[ah(x)+b]^{c-1}h'(x); \qquad x \in (Q_1, P_1) \tag{3.2}$$

and the corresponding truncated *df* $F(x)$ by

$$1-F(x) = -P_2 - \frac{ah(x)+b}{cah'(x)}f(x), \qquad P_2 = \frac{1-P}{P-Q} \tag{3.3}$$

From (6.2), we have

$$1 = -\frac{(P-Q)[ah(x)+b]^{1-c}}{cah'(x)}f(x). \tag{3.4}$$

Let us define

$$m(x,y) = [ah(y)+b]\frac{\frac{\delta}{\delta y}g(x,y)}{h'(y)} \quad \text{and}$$

$$Z(x,y) = [ah(y)+b]^{1-c} \frac{\frac{\delta}{\delta y} g(x,y)}{h'(y)}.$$

Theorem 3.1 (Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\cdots$

$$\begin{aligned} &E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\} \\ &= -\frac{nP_2}{(n-s+1)}\left[E\{g(X_{r:n-1}, X_{s:n-1})\} - E\{g(X_{r:n-1}, X_{s-1:n-1})\}\right] \\ &- \frac{1}{(n-s+1)ca} E\{m(X_{r:n}, X_{s:n})\}. \end{aligned} \tag{3.5}$$

Proof : From Theorem 2.1 and (3.3), we have

$$E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\}$$

$$= \frac{C_{r,s:n}}{(n-s+1)} \iint_{Q_1 \le x < y \le P_1} \frac{\partial}{\partial y} g(x,y)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$
$$\times [1-F(y)]^{n-s}\left[-P_2 - \frac{ah(y)+b}{cah'(y)} f(y)\right] f(x)dxdy$$

and hence the theorem.

Theorem 3.2(Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\ldots$

$$E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\}$$

$$= -\frac{(P-Q)}{(n+1)ca} E\{Z(X_{r:n+1}, X_{s:n+1})\}. \tag{3.6}$$

Proof: Using Theorem 2.1 and (3.4), we can prove this result easily.

To obtain the results for non-truncated distributions, we set $P = 1$ *and* $Q = 0$

For proper choice of a, b, c and $h(x)$, one can get the following distributions:

Table 1 : Examples of distribution $F_1(x) = 1 - [ah(x) + b]^c$; $x \in (\alpha, \beta)$

Distribution function	a	b	c	$h(x)$
1.Power function				
$F_1(x) = \lambda^{-p} x^p; 0 \le x \le \lambda$	$-\lambda^{-p}$	1	1	x^p
	-1	1	1	$\lambda^{-p} x^p$
2.Pareto				
$F_1(x) = 1 - \lambda^p x^{-p}; \lambda \le x < \infty$	λ^p	0	1	x^{-p}
	λ	0	1	x^{-1}
3.Beta of the first kind				
$F_1(x) = 1 - \left(\dfrac{\lambda - x}{\lambda - \beta}\right)^p; \beta \le x \le \lambda$	1	0	p	$\dfrac{\lambda - x}{\lambda - \beta}$
	-1	$\dfrac{\lambda}{\lambda - \beta}$	p	$\dfrac{x}{\lambda - \beta}$
4.Weibull				
$F_1(x) = 1 - e^{-\theta x^p}; 0 \le x < \infty$	1	0	1	$e^{-\theta x^p}$
	1	0	θ	e^{-x^p}
5.Burr type XII				
$F_1(x) = 1 - (1 + \theta x^p)^{-\lambda}; 0 \le x < \infty$	θ	1	$-\lambda$	x^p
	1	0	$-\lambda$	$1 + \theta x^p$

6.Cauchy $F_1(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\theta}{\lambda}\right);$ $-\infty < x < \infty$	$-\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-}\left(\frac{x-\theta}{\lambda}\right)$

3.2. Let us assume that the distribution function $F_1(x)$ is of the general form

$$\mathbf{F_1(x) = [ah(x) + b]^c}, \; x \in (\alpha, \beta) \tag{3.7}$$

where $a \neq 0, b, c \neq 0$ are finite constants and $h(x)$ is continuous, monotonic and differentiable function of x.

Then the truncated *pdf* $f(x)$ is given by

$$f(x) = \frac{ca}{P-Q}[ah(x)+b]^{c-1} h'(x); \qquad x \in (Q_1, P_1) \tag{3.8}$$

and the corresponding truncated *df* $F(x)$ by

$$1 - F(x) = P_3 - \frac{ah(x)+b}{cah'(x)} f(x), \qquad P_3 = \frac{P}{P-Q} \tag{3.9}$$

$$1 = \frac{(P-Q)[ah(x)+b]^{1-c}}{cah'(x)} f(x). \tag{3.10}$$

Theorem 3.3 (Khan & Ali, 1998) For $1 \le r < s \le n, \; n = 1,2,\cdots$

$$\begin{aligned} &E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\} \\ &= \frac{nP_3}{(n-s+1)}\left[E\{g(X_{r:n-1}, X_{s:n-1})\} - E\{g(X_{r:n-1}, X_{s-1:n-1})\}\right] \\ &- \frac{1}{(n-s+1)ca} E\{m(X_{r:n}, X_{s:n})\} \end{aligned} \tag{3.11}$$

Proof : From Theorem 2.1 and (3.9), we can prove this result easily.

Theorem 3.4(Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\cdots$

$$E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\}$$

$$= \frac{(P-Q)}{(n+1)ca} E\{Z(X_{r:n+1}, X_{s:n+1})\}. \tag{3.12}$$

Proof : Using Theorem 2.1 and (3.10), we can prove this result easily.

Examples of some specific distributions contained in (3.7) are given bellow

Table 2 : Examples of the distribution $F_1(x) = [ah(x)+b]^c,\ x \in (\alpha, \beta)$

Distribution function	a	b	c	$h(x)$
1.Power function				
$F_1(x) = \lambda^{-p} x^p; 0 \le x \le \lambda$	λ^{-p}	0	1	x^p
	λ^{-p}	0	p	x
2.Pareto				
$F_1(x) = 1 - \lambda^p x^{-p}; \lambda \le x < \infty$	$-\lambda^p$	1	1	x^{-p}
3.Inverse Weibull				
$F_1(x) = e^{-\theta x^p}; 0 \le x < \infty$	1	0	1	$e^{-\theta x^{-p}}$
	1	0	θ	$e^{-x^{-p}}$

4.Burr type III				
$F_1(x)=(1+\theta x^{-p})^{-\lambda}$; $0 \le x < \infty$	$\begin{cases}\theta \\ 1\end{cases}$	1 1	$-\lambda$ $-\lambda$	x^{-p} θx^{-p}
5.Cauchy				
$F_1(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$; $-\infty < x < \infty$	$\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$

3.3. Suppose the distribution function $F_1(x)$ is of the general form

$$\mathbf{F_1(x)=1-be^{-ah(x)}}, \qquad x\in(\alpha,\beta) \tag{3.13}$$

where $a \neq 0, b > 0$ are constants and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha,\beta]$ Then the truncated *pdf* $f(x)$ is given by

$$f(x)=\frac{ab}{P-Q}e^{-ah(x)}h'(x); \qquad x\in(Q_1,P_1) \tag{3.14}$$

and the corresponding truncated *df* $F(x)$ by

$$1-F(x)=-P_2+\frac{1}{ah'(x)}f(x), \qquad P_2=\frac{1-P}{P-Q} \tag{3.15}$$

From (3.14), we have

$$1=\frac{(P-Q)}{abh'(x)}e^{ah(x)}f(x) \tag{3.16}$$

Let us define

$$W(x,y)=\frac{\frac{\delta}{\delta y}g(x,y)}{h'(y)} \text{ and } T(x,y)=e^{ah(y)}\frac{\frac{\delta}{\delta y}g(x,y)}{h'(y)}.$$

Theorem 3.5 (Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\ldots$

$$\begin{aligned} &E\{g(X_{r:n},X_{s:n})\}-E\{g(X_{r:n},X_{s-1:n})\} \\ &=-\frac{nP_2}{(n-s+1)}\left[E\{g(X_{r:n-1},X_{s:n-1})\}-E\{g(X_{r:n-1},X_{s-1:n-1})\}\right] \\ &-\frac{1}{(n-s+1)a}E\{W(X_{r:n},X_{s:n})\} \end{aligned} \tag{3.17}$$

Proof : In view of Theorem 2.1 and (3.15), we can prove this easily.

Theorem 3.6(Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\ldots$

$$E\{g(X_{r:n},X_{s:n})\}-E\{g(X_{r:n},X_{s-1:n})\}$$

$$=\frac{(P-Q)}{(n+1)ab}E\{T(X_{r:n+1},X_{s:n+1})\} \tag{3.18}$$

Proof: The proof is obvious.

Some well-known specific distributions contained in distribution (3.13) are given as follows:

Table 3: Examples of the distribution $F_1(x)=1-be^{-ah(x)},\ x\in(\alpha,\beta)$

Distribution function	a	b	$h(x)$
1.Power function $F_1(x)=\lambda^{-p}x^p; 0\le x\le\lambda$	-1	1	$\ln(1-\lambda^{-p}x^p)$

2.Pareto			
$F_1(x)=1-\lambda^p x^{-p}; \lambda \le x < \infty$	$\begin{cases} p \\ 1 \end{cases}$	λ^p λ^p	$\ln x$ $p\ln x$
3.Beta of the first kind			
$F_1(x)=1-\left(\dfrac{\lambda-x}{\lambda-\beta}\right)^p; \beta \le x \le \lambda$	$\begin{cases} -p \\ -1 \end{cases}$	1 1	$\ln\dfrac{\lambda-x}{\lambda-\beta}$ $p\ln\left(\dfrac{\lambda-x}{\lambda-\beta}\right)$
4.Weibull			
$F_1(x)=1-e^{-\theta x^p}; 0 \le x < \infty$	$\begin{cases} \theta \\ 1 \end{cases}$	1 1	x^p θx^p
5.Burr type XII			
$F_1(x)=1-(1+\theta x^p)^{-\lambda};$ $0 \le x < \infty$	$\begin{cases} \lambda \\ 1 \end{cases}$	1 1	$\ln(1+\theta x^p)$ $\lambda\ln(1+\theta x^p)$

3.4. Suppose the distribution function $F_1(x)$ is of the general form

$$\mathbf{F_1(x) = be^{-h(x)}}; \; x \in (\alpha, \beta) \tag{3.19}$$

where $a \neq 0, b > 0$ are constants and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha, \beta]$. Then the truncated *pdf* $f(x)$ is given by

$$f(x) = -\frac{a^{bq}}{P-Q} e^{-ah(x)} h'(x); \qquad x \in (Q_1, P_1) \tag{3.20}$$

and the corresponding truncated df $F(x)$ by

$$1 - F(x) = P_3 + \frac{1}{ah'(x)} f(x), \qquad P_3 = \frac{P}{P-Q} \tag{3.21}$$

From (3.20), we have

$$1 = -\frac{(P-Q)}{abh'(x)} e^{ah(x)} f(x) \tag{3.22}$$

Theorem 3.7(Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\ldots$

$$\begin{aligned} &E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\} \\ &= \frac{nP_3}{(n-s+1)}\left[E\{g(X_{r:n-1}, X_{s:n-1})\} - E\{g(X_{r:n-1}, X_{s-1:n-1})\}\right] \\ &+ \frac{1}{(n-s+1)a} E\{W(X_{r:n}, X_{s:n})\}. \end{aligned} \tag{3.23}$$

Proof: In view of Theorem 2.1 and (3.21), we can prove this result easily.

Theorem 3.8(Khan & Ali, 1998) For $1 \le r < s \le n,\ n = 1,2,\ldots$

$$E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\} \tag{3.24}$$

Proof: The proof is obvious.

Some well-known specific distributions contained in distribution (3.19)

are given as follows:

Table 4: Examples of the distribution $F_1(x) = be^{-h(x)}$; $x \in (\alpha, \beta)$

Distribution function	a	b	$h(x)$
1.Power function			
$F_1(x) = \lambda^{-p} x^p; 0 \le x \le \lambda$	$\begin{cases} -p \\ -1 \end{cases}$	λ^{-p} λ^{-p}	$\ln x$ $p \ln x$
2.Pareto			
$F_1(x) = 1 - \lambda^p x^{-p}; \lambda \le x < \infty$	-1	1	$\ln(1 - \lambda^p x^{-p})$
3.Inverse Weibull			
$F_1(x) = e^{-\theta x^p}; 0 \le x < \infty$	θ	1	x^{-p}
4.Burr type III			
$F_1(x) = (1 + \theta x^{-p})^{-\lambda}$; $0 \le x < \infty$	λ	1	$\ln(1 + \theta x^{-p})$

Remarks (i) The recurrence relations between the product moments, moments of ratio of two order statistics, quasi-ranges, joint moment generating functions and joint characteristic functions, whenever they exists can be obtained by setting respectively $g(x, y)$ equal to $x^j y^k, x^j y^{-k}$ $, y - x, e^{t_1 x + t_2 y}$ *and* $e^{it_1 x + it_2 y}$.

(ii) A list of distributions are given in different tables. Further, we can get Rectangular distribution by putting $p=1$ in Beta of first kind distribution. Similarly for Reyleigh and Exponential distributions we put respectively $p=2\, and\, p=1$ in Weibull and for Lomax and Log-logistic put respectively $p=1\, and\, \lambda=1$ in Burr type XII distribution.

(iii) Some recurrence relations on moments of order statistics available in in Khan $et\,al.$ (1983a,b), Balakrishnan $et\,al.$(1988), Khan & Khan (1987),Balakrishnan & Joshi (1981) and Balakrishnan & Malik (1986) can be obtained from our general results by putting $g(x,y)=x^j y^k$.

(iv) At $r=0,$ the Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 6.8 reduce to the general recurrence relations for the expectation of function of single order statistics which on using (2.12) reduces to relations available in Khan & Ali (1997).

4. An Identity for the Product Moments of Order Statistics:

Mohie El-Din *et al.* (1996) have obtained a general identity for product moments of order statistics in a class of distribution functions, including Pareto, Weibull, exponential, Rayleigh and Burr distributions. Several recurrence relations are particular cases of the theorem stated.

Theorem 1 (Mohie El-Din *et al.*, 1996). Let $\alpha \in \Re, \beta \geq 0$ with $\alpha+\beta \neq 0,$ let $h(t)$ be a function on $(0,1)$ with

$$\frac{d}{dt}h(t)=\frac{1}{d}(1-t)^{q-1}, \; d>0, \; q \in \mathrm{Z}, \tag{4.1}$$

Let F be given by

$$F^{-1}(t)=\begin{cases}\exp\{h(t)\}, & \beta=0\\ \{\beta h(t)\}^{1/\beta}, & \beta>0\end{cases} \tag{4.2}$$

where, $\{\beta h(t)\}^{1/\beta}\in\Re$, and let

$$-\infty<E(X^{j}_{r:n}X^{\alpha+\beta}_{s:n}),E(X^{j}_{r:n}X^{\alpha+\beta}_{s-1:n}),E(X^{j}_{r:n+q}X^{\alpha}_{s:n+q})<\infty,\quad j>0,$$

for some integers r,s and n with $1\le r<s\le\min(n,n+q)$, $n+q\ge 2$. Then the recurrence relation

$$E(X^{j}_{r:n}X^{\alpha+\beta}_{s:n})-E(X^{j}_{r:n}X^{\alpha+\beta}_{s-1:n})=(\alpha+\beta)C(nq,s)E(X^{j}_{r:n+q}X^{\alpha}_{s:n+q}),$$

$$1\le r<s\le n+q \tag{4.33}$$

is valid with,

$$C(n,q,s)=\frac{n!(n+q-s)!}{d(n-s+1)!(n+q)!}.$$

This theorem corresponds to Theorem 1 in Kamps (1991) in the single order statistics case.

For different values of q,β and c, we have the following table

q, β	$F(x)$	$x \in$	Distribution	$(\alpha+\beta)C(n,q,s)$
0,0	$1-e^{cd}x^{-d},\ c \in \Re$	(e^{c},∞)	Pareto	$\frac{\alpha}{d(n-s+1)}$
0,> 0	$1-\exp\left\{-d\left(\frac{1}{\beta}x^{\beta}-c\right)\right\},$ $c \geq 0$	$((\beta c)^{1/\beta},\infty)$	Weibull	$\frac{\alpha+\beta}{d(n-s+1)}$
0,1	$1-\exp\{-d(x-c)\}$	(c,∞)	exponential	$\frac{\alpha+1}{d(n-s+1)}$
0,2	$1-\exp\left\{-d\left(\frac{x^{2}}{2}-c\right)\right\}$	$((2c)^{1/2},\infty)$	Rayleigh	$\frac{\alpha+2}{d(n-s+1)}$
<0, >0	$1-\left\{dq\left(c-\frac{1}{\beta}x^{\beta}\right)\right\}^{1/q}$	$\left(\left(c-\frac{1}{dq}\right)^{1/\beta},\infty\right)$	Burr	$\frac{n!(\alpha+\beta)(n+q-s)!}{d(n-s+1)!(n+q)!}$
>0,0	$1-\{dq(c-\log x)\}^{1/q}$	$(e^{c-(1/qd)},e^{c})$		
<0,0	$1-\{dq(c-\log x)\}^{1/q}$	$(e^{c-(1/qd)},\infty)$		

Chapter 4

Recurrence Relations and Identities of Order Statistics from Independent and Non-identically Distributed Random Variables

1. Introduction:

In this chapter we have discussed the recurrence relations and identities of product moments for non-identically distributed random variables.

In section 2, some recurrence relations derived by Balakrishnan (1988)are given. In section 3 some recurrence relations and identities derived by Beg (1991) are given.

By assuming that $X_1, X_2, \ldots, X_n$ are independent variates with $X_i (i=1,2,\ldots,n)$ having *pdf* $f_i(x)$ and *df* $F_i(x)$, Vaughan & Venables (1972) have shown that the density function of $X_{r:n} (1 \le r \le n)$ can be written down as

$$h_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} per \begin{array}{c} \begin{bmatrix} F_1(x) & 1-F_1(x) & f_1(x) \\ \vdots & \vdots & \vdots \\ F_n(x) & 1-F_n(x) & f_n(x) \end{bmatrix} \\ \begin{matrix} \quad r-1 \quad & \quad n-r \quad & \quad 1 \quad \end{matrix} \end{array} \tag{1.1}$$

where $per[A]$ denotes the permanent of a square matrix A; the permanent is defined just like the determinant, except that all signs in the expansion are positive. Vaughan & Venables (1972) have also shown that the joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ can be written down as

$$h_{r,s:n}(x,y)=\frac{1}{(r-1)!(s-r-1)!(n-s)!}$$

$$\times\, per\begin{array}{c}\begin{bmatrix} F_1(x) & F_1(y)-F_1(x) & 1-F_1(y) & f_1(x) & f_1(y)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ F_n(x) & F_n(y)-F_n(x) & 1-F_n(y) & f_n(x) & f_n(y)\end{bmatrix}\\ \begin{array}{ccccc} r-1 & s-r-1 & n-s & 1 & 1\end{array}\end{array}$$

$$-\infty<x<y<\infty \tag{1.2}$$

2. Recurrence Relations for Order Statistics from n Independently and Non-identically Distributed Random Variables:

Balakrishnan (1988) has generalized the various recurrence relations of order statistics to the case when these order statistics are obtained from n independent and non-identically distributed random variables.

He used $h_{r:n-m}^{[i_1,i_2,\ldots,i_m]}(x), 1\le r\le n-m,$ to denote the density function of the rth order statistics in a sample of size $n-m$ obtained by dropping $X_{i_1},X_{i_2},\ldots,X_{i_m}$ from the original set of n variables. We then have the following relation.

Relation 1. (Balakrishnan, 1988) For $1\le r\le n-1$,

$$r\,h_{r+1:n}(x)+(n-r)h_{r:n}(x)=\sum_{i=1}^{n}h_{r:n-1}^{[i]}(x). \tag{2.1}$$

Proof. First, consider the permanent expression of $r\,h_{r+1:n}(x)$ from (1.1). Upon expanding this permanent by its first row, we find

$$r\,h_{r+1:n}(x)=\sum_{i=1}^{n}F_i(x)h_{r:n-1}^{[i]}(x). \tag{2.2}$$

Next, consider the expression of $(n-r)h_{r:n}(x)$ from equation (2.1). Upon expanding this permanent by its last row, we get

$$(n-r)h_{r:n}(x) = \sum_{i=1}^{n} \{1 - F_i(x)\} h_{r:n-1}^{[i]}(x). \tag{2.3}$$

Relation (2.1) follows immediately upon adding equations (2.2) and (2.3).

Let us now denote

$$S_{1:n-m}(x) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} h_{1:n-m}^{[i_1, i_2, \ldots, i_m]}(x)$$

and

$$S_{n-m:n-m}(x) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} h_{1:n-m}^{[i_1, i_2, \ldots, i_m]}(x),$$

with $S_{1:n}(x) \equiv h_{1:n}(x)$ and $S_{n:n}(x) \equiv h_{n:n}(x)$. Then by repeated application of Relation 1, Balakrishnan (1988) directly obtain the following relations.

Relation 2. (Balakrishnan, 1988) For $1 \le r \le n-1$,

$$h_{r:n}(x) = \sum_{i=1}^{n} (-1)^{j-r} \binom{j-1}{r-1} S_{j:j}(x). \tag{2.4}$$

Relation3. (Balakrishnan, 1988) For $2 \le r \le n$,

$$h_{r:n}(x) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{r-1} S_{1:j}(x). \tag{2.5}$$

Remark 2.1. For the case when the X_i's are identically distributed, it is easy to see that

$$S_{j:j}(x) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} h_{j:j}^{[i_1, i_2, \ldots, i_m]}(x) = \binom{n}{j} h_{j:j}(x)$$

and

$$S_{1:j}(x) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} h_{1:j}^{[i_1, i_2, \ldots, i_m]}(x) = \binom{n}{j} h_{1:j}(x).$$

As a result, Relation 2 and 3 yield (in terms of moments)

$$\mu_{r:n}^{(k)} = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \binom{n}{j} \mu_{j:j}^{(k)}, \quad 1 \le r \le n-1, \qquad \text{(Srikanthan,1962)}$$

and

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{r-1} \binom{n}{j} \mu_{1:j}^{(k)}, \quad 2 \le r \le n. \qquad \text{(Srikanthan,1962)}$$

Now using $h_{r,s:n-1}^{[i]}(x, y), 1 \le r < s \le n,$ to denote the joint density function of the rth and sth order statistics in a sample of size $n-1$ obtained by dropping X_i from the original set of n variables, we have the following recurrence relation.

Relation 4. (Balakrishnan, 1988) For $2 \le r < s \le n,$

$$(r-1)h_{r,s:n}(x, y) + (s-r)h_{r-1,s:n}(x, y) + (n-s+1)h_{r-1,s-1:n}(x, y)$$

$$= \sum_{i=1}^{n} h_{r-1,s-1:n-1}^{[i]}(x, y). \qquad (2.6)$$

Proof. Expanding the permanent in (1.2) by its first, rth and last row respectively, we find

$$(r-1)h_{r,s:n}(x,y)=\sum_{i=1}^{n}F_i(x)h_{r-1,s-1:n-1}^{[i]}(x,y), \tag{2.7}$$

$$(s-r)h_{r-1,s:n}(x,y)=\sum_{i=1}^{n}\{F_i(y)-F_i(x)\}h_{r-1,s-1:n-1}^{[i]}(x,y), \tag{2.8}$$

$$(n-s+1)h_{r-1,s:n}(x,y)=\sum_{i=1}^{n}\{1-F_i(y)\}h_{r-1,s-1:n-1}^{[i]}(x,y). \tag{2.9}$$

Relation (2.6) follows upon adding equations (2.7),(2.8) and (2.9).

Remark 2.2. For the p outlier model, that is, $F_1=F_2=\cdots=F_{n-p}=F$ and $F_{n-p+1}=\cdots=F_n=G$, relations 1 and 4,resoectively yield

$$r\,h_{r+1:n}(x)+(n-r)h_{r:n}(x)=(n-p)h_{r:n-1}^{[F]}(x)+p\,h_{r:n-1}^{[G]}(x)$$

and

$$(r-1)h_{r,s:n}(x,y)+(s-r)h_{r-1,s:n}(x,y)+(n-s+1)h_{r-1,s-1:n}(x,y)$$

$$=(n-p)h_{r-1,s-1:n-1}^{[F]}(x,y)+p\,h_{r-1,s-1:n-1}^{[G]}(x,y),$$

where $h_{r:n-1}^{[F]}(x)$ and $h_{r:n-1}^{[G]}(x)$ are the density functions of the rth order statistics in a sample of size $n-1$ from the p outlier model and the $(p-1)$ outlier model, respectively.

3. Recurrence Relations and Identities for Product Moments:

Beg (1991) have obtained the recurrence relations and identities for product moments of order statistics corresponding to non-identically distributed variables as follows:

Suppose $g_1(x)$ and $g_2(x)$ are Borel measurable functions from $\Re$ to $\Re$. Then the single and product moments of functions of order statistics can be obtained from (1.1) and (1.2) as

$$E\{g_1(X_{r:n})\} = \int_{-\infty}^{\infty} g_1(x) h_{r:n}(x) dx, \ 1 \le r \le n \tag{3.1}$$

and

$$E\{g_1(X_{r:n}) g_2(X_{s:n})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} g_1(x) g_2(y) h_{r,s:n}(x,y) dxdy, \ 1 \le r < s \le n \tag{3.2}$$

We assume that all these moments exist.

The following notation will be used here. If $S \subset N = \{1,2,\cdots,n\}$ then S' will denote the compliment of S in N and $|S|$ will denote the cardinality of S. Let $X_{r:s}$ denote the rth order statistic for X_i, $i \in S$.For convenience, for fixed x, F will denote the column $(F_1(x), F_2(x), \cdots, F_n(x))'$, f the column vector $(f_1(x), f_2(x), \cdots, f_n(x))'$ and 1 the column vectors of all ones. We will denote by $A[S|.)$ the matrix obtained from A by taking rows whose indices are in S.

Joshi & Balakrishnan (1982) obtained several recurrence relations and identities for product moments of order statistics from an arbitrary continuous distribution when all X_i's are independent and identically distributed. Here, we generalize these results when all X_i's are independent but not assumed to be identically distributed. Arnold & Balakrishnan (1989) have given a comprehensive collection of such recurrence relations.

3.1. Recurrence Relations:

Theorem 3.1. (Beg, 1991) For $1 \le r < s \le n,$

$$E\{g_1(X_{r:n})g_2(X_{s:n})$$

$$+\sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{j+k}{k}\sum_{|S|=n-j-k}E\{g_2(X_{n-s-j+1:s})g_1(X_{n-r-j+1:s})\}$$

$$=\sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\sum_{|S|=s-i}E\{g_1(X_{s-i:s})g_2(X_{i:S'})\} \qquad (3.3)$$

Proof. Consider $I=[(r-1)!(s-r-1)!(n-s)!]^{-1}.J$, where

$$J=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y)per[\underset{r-1}{F(x)}\quad \underset{s-r-1}{F(y)-F(x)}\quad \underset{n-s}{1-F(y)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}]dx\,dy$$

$$=\sum_{t=0}^{s-r-1}(-1)^{s-r-t-1}\binom{s-r-1}{t}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y)per[\underset{s-t-2}{F(x)}\quad \underset{1}{f(x)}\quad \underset{t}{F(y)}\quad \underset{n-s}{1-F(y)}\quad \underset{1}{f(y)}]dx\,dy$$

$$=\sum_{t=0}^{s-r-1}(-1)^{s-r-t-1}\binom{s-r-1}{t}$$

$$\times\sum_{|S|=s-t-1}\int_{-\infty}^{\infty} g_1(x)per[\underset{s-t-2}{F(x)}\quad \underset{1}{f(x)}][S|.)dx$$

$$\times \int_{-\infty}^{\infty} g_2(y) per[\underset{t}{F(y)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(y)}][S'|.)dy$$

$$= \sum_{t=0}^{s-r-1} (-1)^{s-r-t-1} \binom{s-r-1}{t} \sum_{|S|=s-t-1} (s-t-2)!t!(n-s)!E\{g_1(X_{s-t-1:S})\}$$

$$\times E\{g_2(X_{t+1:S'})\}, \text{ using (3.1).}$$

Writing $i = t+1$, we find

$$J = \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-r-1}{i-1} \sum_{|S|=s-i} (s-i-1)!(i-1)!(n-s)!E\{g_1(X_{s-i:S})\}$$

$$\times E\{g_2(X_{i:S'})\},$$

and

$$I = \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-i-1}{r-1} \sum_{|S|=s-i} E\{g_1(X_{s-i:S})\}E\{g_2(X_{i:S'})\},$$

which is the right hand side of (3.3). Further

$$I = [(r-1)!(s-r-1)!(n-s)!]^{-1}$$

$$\times [\iint_{x<y} g_1(x)g_2(y) per[\underset{r-1}{F(x)}\ \ \underset{s-r-1}{F(y)-F(x)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy$$

$$\iint_{y<x} g_1(x)g_2(y) per[\underset{r-1}{F(x)}\ \ \underset{s-r-1}{F(y)-F(x)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy]$$

$$= E\{g_1(X_{r:n})g_2(X_{s:n})\} + [(r-1)!(s-r-1)!(n-s)!]^{-1} J_1,$$

where

$$J_1 = \iint\limits_{y<x} g_1(x)g_2(y)\,per[\underset{r-1}{F(x)}\ \ \underset{s-r-1}{F(y)-F(x)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy$$

$$= (-1)^{s-r-1} \iint\limits_{y<x} g_1(x)g_2(y)$$

$$\times\, per[\underset{r-1}{1-\{1-F(x)\}}\ \ \underset{s-r-1}{F(x)-F(y)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy$$

$$= \sum_{k=0}^{r-1}(-1)^{s-k-2}\binom{r-1}{k}\iint\limits_{y<x} g_1(x)g_2(y)$$

$$\times\, per[\underset{k}{1}\ \ \underset{r-k-1}{1-F(x)}\ \ \underset{s-r-1}{F(x)-F(y)}\ \ \underset{n-s}{1-F(y)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy$$

$$= \sum_{k=0}^{r-1}(-1)^{s-k-2}\binom{r-1}{k}\sum_{j=0}^{n-s}(-1)^{n-s-j}\binom{n-s}{j}\iint\limits_{y<x} g_1(x)g_2(y)$$

$$\times\, per[\underset{j+k}{1}\ \ \underset{n-s-j}{F(y)}\ \ \underset{s-r-1}{F(x)-F(y)}\ \ \underset{r-k-1}{1-F(x)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}]dx\,dy$$

$$= \sum_{k=0}^{r-1}\sum_{j=0}^{n-s}(-1)^{n-j-k}\binom{r-1}{k}\binom{n-s}{j}\sum_{|S|=n-j-k}(j+k)!\iint\limits_{y<x} g_1(x)g_2(y)$$

$$\times\, per[\underset{n-s-j}{F(y)}\ \ \underset{s-r-1}{F(x)-F(y)}\ \ \underset{r-k-1}{1-F(x)}\ \ \underset{1}{f(x)}\ \ \underset{1}{f(y)}][S|.)dx\,dy$$

$$= \sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{r-1}{k}\binom{n-s}{j}(j+k)!$$

$$\times \sum_{|S|=n-j-k} E\{g_2(X_{n-s-j+1:S})g_1(X_{n-r-j+1:S})\}$$

$$\times (n-s-j)!(s-r-1)!(r-k-1)!, \qquad \text{(Using (3.2))}$$

On simplification, we see that I is also equal to the left hand side of (3.3).

Corollary 3.1. If $g_2(x)=1$, theorem 3.1 yields

$$E\{g_1(X_{r:n})+\sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{j+k}{k}\sum_{|S|=n-j-k}E\{g_1(X_{n-r-j+1:s})\}$$

$$=\sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\sum_{|S|=s-i}E\{g_1(X_{s-i:s})\}. \tag{3.4}$$

which is a recurrence relation involving single moments of functions of order statistics.

Corollary 3.2. For the case of a sample of n independent and identically distributed random variables $X_1, X_2, \cdots X_n$ having pdf $f(x)$ and cdf $F(x)$

Theorem 3.1 simply reduces to

$$E\{g_1(X_{r:n})g_2(X_{s:n})+\sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{k}$$

$$\times E\{g_2(X_{n-s-j+1:n-j-k})g_1(X_{n-r-j+1:n-j-k})\}$$

$$=\sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\binom{n}{s-i}E\{g_1(X_{s-i:s-i})g_2(X_{i:n-s+i})\} \tag{3.5}$$

Taking $g_1(x)=g_2(x)=x$ and writing $\mu_{r:n}=E(X_{r:n})$ and $\mu_{r,s:n}=E(X_{r:n}X_{s:n})$, (3.5) reduces to Theorem 2.1 of Joshi & Balakrishnan (1982).

Corollary 3.3. For the $p-outlier$ model that is, $F_1=F_2=\cdots=F_{n-p}=F$ and $F_{n-p+1}=\cdots=F_n=G$ $(outlier\ distributuion)$ Theorem 3.1 yields

$$E\{g_1(X_{r:n})g_2(X_{s:n})$$

$$+\sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j+k}\binom{j+k}{k}\sum_{a=0}^{p}\binom{p}{a}\binom{n-p}{n-j-k-a}$$

$$\times E\{g_2(X_{n-s-j+1:n-j-k,a})g_1(X_{n-r-j+1:n-j-k,a})\}$$

$$=\sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\sum_{a=0}^{p}\binom{p}{a}\binom{n-p}{s-i-a}E\{g_1(X_{s-i:s-i,a})\}$$

$$\times E\{g_2(X_{i:n-s+i,p-s})\}$$

where $X_{r:n,a}$ denotes the rth order statistics from a sample of size n of which 'a' are outliers.

3.2. Some identities

Theorem 3.2. (Beg, 1991) For $1\le i\le n-2$,

$$\sum_{|S|=n-i} i!E\{g_1(X_{n-i-1:S})g_2(X_{n-i:S})\}$$

$$=\frac{1}{(n-i-2)!}\sum_{j=0}^{i}\sum_{k=0}^{i}\binom{i}{j}j!(n-j-2)!\ E\{g_1(X_{n-j-1:n})g_2(X_{n-k:n})\} \quad (3.6)$$

Proof. $\sum\limits_{|S|=n-i} i!E\{g_1(X_{n-i-1:S})g_2(X_{n-i:S})\}$

$$= \sum_{|S|=n-i} \frac{i!}{(n-i-2)!} \iint\limits_{x<y} g_1(x)g_2(y)per[\underset{n-i-2}{F(x)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}][S|.)dx\,dy$$

$$= \frac{1}{(n-i-2)!} \iint\limits_{x<y} g_1(x)g_2(y)per[\underset{i}{1}\quad \underset{n-i-2}{F(x)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}]dx\,dy$$

$$= \frac{1}{(n-i-2)!} \iint\limits_{x<y} g_1(x)g_2(y)$$

$$\times\, per[\underset{i}{F(x)+\{F(y)-F(x)\}+\{1-F(y)\}}\quad \underset{n-i-2}{F(x)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}]dx\,dy$$

$$= \frac{\sum\limits_{j=0}^{i}\binom{i}{j}}{(n-i-2)!} \iint\limits_{x<y} g_1(x)g_2(y)$$

$$\times\, per[\underset{j}{\{F(y)-F(x)\}+\{1-F(y)\}}\quad \underset{n-j-2}{F(x)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}]dx\,dy$$

$$= \frac{\sum\limits_{j=0}^{i}\binom{i}{j}\sum\limits_{k=0}^{j}\binom{j}{k}}{(n-i-2)!} \iint\limits_{x<y} g_1(x)g_2(y)$$

$$\times\, per[\underset{n-j-2}{F(x)}\quad \underset{j-k}{F(y)-F(x)}\quad \underset{k}{1-F(y)}\quad \underset{1}{f(x)}\quad \underset{1}{f(y)}]dx\,dy$$

$$= \frac{\sum\limits_{j=o}^{i}\sum\limits_{k=0}^{j}\binom{i}{j}\binom{j}{k}}{(n-i-2)!} E\{g_1(X_{n-j-1:n})g_2(X_{n-k:n})\}(n-j-2)!(j-k)!k!$$

$$= \frac{\sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} j!(n-j-2)!}{(n-i-2)!} E\{g_1(X_{n-j-1:n}) g_2(X_{n-k:n})\}.$$

This completes the proof.

Corresponding to the corollaries 3.1,3.2 and 3.3, we have the following

Corollary 3.4. $\sum_{|S|=n-i} i! E\{g_1(X_{n-i-1:S})\}$

$$= \frac{\sum_{j=o}^{i} \sum_{k=0}^{j} \binom{i}{j} j!(n-j-2)!}{(n-i-2)!} E\{g_1(X_{n-j-1:n})\} \tag{3.7}$$

Corollary 3.5. $(n-i)(n-i-1) \sum_{j=0}^{i} \left[\binom{i}{j} \Big/ \binom{n-2}{j} \right]$

$$\times \sum_{k=0}^{j} E\{g_1(X_{n-j-1:n}) g_2(X_{n-k:n})\}$$

$$= n(n-1) E\{g_1(X_{n-i-1:n-i}) g_2(X_{n-i:n-i})\}. \tag{3.8}$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r,s:n} = E(X_{r:n} X_{s:n})$, (3.8) reduces to Theorem 3.1 of Joshi & Balakrishnan (1982).

Corollary 3.6. $i! \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{n-i-a} E\{g_1(X_{n-i-1:n-i,a}) g_2(X_{n-i:n-1,a})\}$

$$= \frac{\sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} j!(n-j-2)!}{(n-i-2)!} E\{g_1(X_{n-j-1:n}) g_2(X_{n-k:n})\}. \tag{3.9}$$

Theorem 3.3. (Beg, 1991) For $1 \le r < n$,

$$\sum_{r=1}^{n-1} E\{g_1(X_{r:n})g_2(X_{r+1:n})\} + \sum_{j=2}^{n} \sum_{|S|=j} E\{g_2(X_{1:S})g_1(X_{j:S})\}$$

$$= \sum_{j=1}^{n-1} \sum_{|S|=j} E\{g_1(X_{j:S})g_2(X_{1:S'})\} \tag{3.10}$$

Proof. Consider the sum of integrals

$$I = \sum_{j=1}^{n-1} \frac{1}{(j-1)!(n-j-1)!}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y) per[\underset{j-1}{F(x)} \; \underset{n-j-1}{1-F(y)} \; \underset{1}{f(x)} \; \underset{1}{f(y)}]dx\,dy$$

$$= \sum_{j=1}^{n-1} \frac{1}{(j-1)!(n-j-1)!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y)$$

$$\times \sum_{|S|=j} per[\underset{j-1}{F(x)} \; \underset{1}{f(x)}][S|.)\, per[\underset{n-j-1}{1-F(y)} \; \underset{1}{f(y)}][S'|.)dx\,dy$$

$$= \sum_{j=1}^{n-1} \frac{1}{(j-1)!(n-j-1)!} \sum_{|S|=j} \int_{-\infty}^{\infty} g_1(x) per[\underset{j-1}{F(x)} \; \underset{1}{f(x)}][S|.)dx$$

$$\times \int_{-\infty}^{\infty} g_2(y) per[\underset{n-j-1}{1-F(y)} \; \underset{1}{f(y)}][S'|.)dy$$

$$= \sum_{j=1}^{n-1} \sum_{|S|=j} E\{g_1(X_{j:S})\}E\{g_2(X_{1:S'})\}$$

which is the right hand side of (3.10). Further, we can write

$$I=\sum_{j=1}^{n-1}\frac{1}{(j-1)!(n-j-1)!}$$

$$\times\Big[\iint\limits_{x<y} g_1(x)g_2(y)\,per[\underset{j-1}{F(x)}\ \underset{n-j-1}{1-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy$$

$$+\iint\limits_{y<x} g_1(x)g_2(y)\,per[\underset{j-1}{F(x)}\ \underset{n-j-1}{1-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy\Big]$$

$$=\sum_{j=1}^{n-1}E\{g_1(X_{j:n})g_2(X_{j+1:n})\}+J,$$

where

$$J=\sum_{j=1}^{n-1}\frac{1}{(j-1)!(n-j-1)!}$$

$$\times\iint\limits_{y<x} g_1(x)g_2(y)\,per[\underset{j-1}{F(x)}\ \underset{n-j-1}{1-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy$$

$$=\frac{1}{(n-2)!}\iint\limits_{y<x} g_1(x)g_2(y)\sum_{j=1}^{n-1}\binom{n-2}{j-1}per[\underset{j-1}{F(x)}\ \underset{n-j-1}{1-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy$$

$$=\frac{1}{(n-2)!}\iint\limits_{y<x} g_1(x)g_2(y)\,per[\underset{n-2}{F(x)+1-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy$$

$$=\frac{1}{(n-2)!}\iint\limits_{y<x} g_1(x)g_2(y)\sum_{k=0}^{n-2}\binom{n-2}{k}per[\underset{k}{1}\ \underset{n-k-1}{F(x)-F(y)}\ \underset{1}{f(x)}\ \underset{1}{f(y)}]dx\,dy$$

$$= \frac{1}{(n-2)!} \sum_{k=0}^{n-2} \binom{n-2}{k} \sum_{|S|=n-k} k! \iint_{y<x} g_1(x) g_2(y)$$

$$\times\, per[\underset{n-k-2}{F(x)-F(y)} \;\; \underset{1}{f(x)} \;\; \underset{1}{f(y)}][S|.) dx\, dy$$

$$= \frac{1}{(n-2)!} \sum_{k=0}^{n-2} \binom{n-2}{k} k! \sum_{|S|=n-k} E\{g_2(X_{1:S}) g_1(X_{n-k:S})\}(n-k-2)!.$$

Writing $j = n-k$, we get

$$J = \sum_{j=2}^{n} \sum_{|S|=j} E\{g_2(X_{1:S}) g_1(X_{j:S})\}$$

and we see that I is also equal to the left hand side of (3.10). This completes the proof of Theorem 3.3.

Corresponding to the corollaries 3.1,3.2 and 3.3, we have the following.

Corollary 3.7. $\sum_{r=1}^{n-1} E\{g_1(X_{r:n})\} + \sum_{j=2}^{n} \sum_{|S|=j} E\{g_1(X_{j:S})\}$

$$= \sum_{j=1}^{n-1} \sum_{|S|=j} E\{g_1(X_{j:S})\} \tag{3.11}$$

Corollary 3.7.

$$\sum_{r=1}^{n-1} E\{g_1(X_{r:n}) g_2(X_{r+1:n})\} + \sum_{j=2}^{n} \binom{n}{j} E\{g_2(X_{1:j}) g_1(X_{j:j})\}$$

$$= \sum_{j=1}^{n-1} \binom{n}{j} E\{g_1(X_{j:j}) g_2(X_{1:n-j})\} \tag{3.12}$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r:n} = E(X_{r:n})$ and $\mu_{r,s:n} = E(X_{r:n}X_{s:n})$, (3.12) reduces to Theorem 3.2 of Joshi & Balakrishnan (1982).

Corollary 3.8. $\sum_{r=1}^{n-1} E\{g_1(X_{r:n})g_2(X_{r+1:n})\}$

$$+\sum_{j=2}^{n}\sum_{a=0}^{p}\binom{p}{a}\binom{n-p}{j-a}E\{g_2(X_{1:j,a})g_1(X_{j:j,a})\}$$

$$=\sum_{j=1}^{n-1}\sum_{a=0}^{p}\binom{p}{a}\binom{n-p}{j-a}E\{g_1(X_{j:j,a})g_2(X_{1:n-j,p-a})\} \tag{3.13}$$

Theorem 3.4. (Beg, 1991) $1 \le r < s \le n$ *and* $1 \le k \le n-r$,

$$\sum_{s=r+1}^{n-k+1}\binom{n-s}{k-1}E\{g_1(X_{r:n})g_2(X_{s:n})\}$$

$$+\sum_{i=1}^{r}\sum_{s=r+1}^{r+k}\binom{s-i-1}{s-r-1}\binom{n-s}{n-k-r}E\{g_2(X_{i:n})g_1(X_{s:n})\}$$

$$= \sum_{|S|=n-k} E\{g_1(X_{r:S})g_2(X_{1:S'})\} \tag{3.14}$$

The proof of this theorem is very similar to that of Theorem 3.2 and 3.3.

Corresponding to the corollaries 3.1,3.2 and 3.3, we have the following.

Corollary 3.9. $\sum_{s=r+1}^{n-k+1}\binom{n-s}{k-1}E\{g_1(X_{r:n})\}$

$$+\sum_{i=1}^{r}\sum_{s=r+1}^{r+k}\binom{s-i-1}{s-r-1}\binom{n-s}{n-k-r}E\{g_1(X_{s:n})\}$$

$$= \sum_{|S|=n-k} E\{g_1(X_{r:S})\}. \tag{3.15}$$

Corollary 3.10. $\displaystyle\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\{g_1(X_{r:n})g_2(X_{s:n})\}$

$$+\sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1}\binom{n-s}{n-k-r} E\{g_2(X_{i:n})g_1(X_{s:n})\}$$

$$=\binom{n}{n-k} E\{g_1(X_{r:n-k})g_2(X_{1:k})\} \tag{3.16}$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r:n} = E(X_{r:n})$ and $\mu_{r,s:n} = E(X_{r:n}X_{s:n})$, (3.16) reduces to Theorem 3.3 of Joshi & Balakrishnan (1982).

Corollary 3.11. $\displaystyle\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\{g_1(X_{r:n})g_2(X_{s:n})\}$

$$+\sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1}\binom{n-s}{n-k-r} E\{g_2(X_{i:n})g_1(X_{s:n})\}$$

$$= \sum_{a=0}^{p} \binom{p}{a}\binom{n-p}{n-k-a} E\{g_1(X_{r:n-k,a})g_2(X_{1:k,p-a})\} \tag{3.17}$$

Remark.3.1 It may be remarked that for suitable $g_1(x)$ and $g_2(x)$, Theorems 3.1,3.2,3.3 and 3.4 yield recurrence relations and identities involving moment generating functions and distribution functions etc.

Chapter 5

Operator and Indicator Methods in Order Statistics

1. Introduction:

In the last thirty years or so, many identities and recurrence relations for distributions of order statistics have been established by several authors including Srikanthan (1962), Govindarajulu (1963), Joshi (1973), Arnold (1977), Joshi & Balakrishnan (1982), Khan *et al.* (1983a,b) and Balakrishnan & Malik (1985) *etc.* The proofs for the identities of order statistics established by these and other authors hinge upon some combinatorial techniques, some easy and some involved.

In this chapter we discussed operator and indicator methods in order statistics. In section 2 operator methods by Balasubramanian *et al.*(1992) is discussed and in section 4 indicator functions of sets are used to prove various recurrence relations for order statistics. This approach, in addition to being simpler, lends itself to easy generalizations for higher orders.

2. Operator Methods for Identities of Order Statistics:

Operator equalities (based on both difference and differential operators) derived by Balasubramannian *et al.* (1992), which when applied on suitably chosen functions generate identities for distributions of order statistics. Many known identities are deduced easily by this approach, and, in addition many new identities are also generated. Furthermore, unlike the direct combinatorial approach, the operator methods discussed in this section may easily be extended to identities involving joint distributions of order statistics. This is illustrated for joint distributions of two order statistics and the author derives many new identities in this case.

2.1. Identities for single order statistics by operator: (Balasubramanian *et al.*, 1992)

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a population with cumulative distribution function $F(x)$ and probability density function $f(x)$. Let $X_{1:n} \le \ldots \le X_{n:n}$ be the order statistics obtained from the above sample. Then, the density function of $X_{r:n}$ $(1 \le r \le n)$ is given by

$$f_{r:n} = \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x) \quad -\infty < x < \infty \tag{2.1}$$

Further, let Δ and E be the difference and shift operators with common difference 1 acting on functions of y (where y is independent of x). Then we have the following operator equalities satisfied by distributions of order statistics.

Theorem 1. (Balasubramannian, *et al.*,1992) For $n \ge 2$,

$$\sum_{r=1}^{n} f_{r:n}(x)E^{n-r} = \sum \binom{n}{r} f_{1:r}(x)\Delta^{r-1} \tag{2.2}$$

and

$$\sum_{r=1}^{n} f_{r:n}(x)E^{r-1} = \sum_{r=1}^{n} \binom{n}{r} f_{r:r}(x)\Delta^{r-1}. \tag{2.3}$$

Proof. We have,

$$\sum_{r=1}^{n} (1+t)^{n-r} f_{r:n}(x)$$

$$= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}[(1+t)\{1-F(x)\}]^{n-r} f(x)$$

$$= n[1 + t\{1 - F(x)\}]^{n-1} f(x)$$

$$= n \sum_{r=0}^{n} \binom{n-1}{r} \{1 - F(x)\}^r f(x) t^r.$$

Then

$$\sum_{r=1}^{n} (1+t)^{n-r} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) t^{r-1} \qquad (2.4)$$

Equation (2.2) follows from (2.4) on setting $t = \Delta$.

Similarly (2.3) can also be proved.

Example 1. For the function $T(y) = \dfrac{1}{n-y}$,

$$E^{n-r} T(y) = \frac{1}{r-y} \; and \; \Delta^{r-1} T(y) = \frac{(r-1)!}{(n-y)^{(r)}},$$

where $m^{(k)} = m(m-1)(m-2)\ldots(m-k+1) = \dfrac{m!}{(m-k)!}$

$for\, k \geq 1\, and = 1\, for\, k = 0.$

Setting $y = 0$, (2.2) now gives

$$\sum_{r=1}^{n} \frac{1}{r} f_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r} f_{1:r}(x).$$

Similarly, (2.3) gives

$$\sum_{r=1}^{n} \frac{1}{n-r+1} f_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r} f_{r:r}(x). \qquad \text{(Joshi, 1973)}$$

Example 2. By choosing the function $T(y) = 1/(N-y)^{(k)}$, from (2.2) and (2.3) (with $y = 0$) the following identities are obtained

$$\sum_{r=1}^{n} \frac{1}{(N-n+r)^{(k)}} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} \frac{(k+r-2)^{(r-1)}}{N^{(k+r-1)}} f_{1:r}(x), \tag{2.5}$$

$$\sum_{r=1}^{n} \frac{1}{(N-n+r)^{(k)}} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} \frac{(k+r-2)^{(r-1)}}{N^{(k+r-1)}} f_{r:r}(x), \tag{2.6}$$

By setting $N = n+k-1$ in these two identities, we get the identities due to Balakrishnan & Malik (1985) and, therefore, (2.5) and (2.6) can be regarded as generalization of these results.

Alternatively, by setting $t = D$ in (2.4) where D is the differentiation operator acting on function of y (with y being independent of x), Balasubramanian *et al.* (1992) have also shown:

Theorem 2. *For* $n \geq 2$,

$$\sum_{r=1}^{n} f_{r:n}(x)(1+D)^{n-r} = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) D^{r-1} \tag{2.7}$$

$$\sum_{r=1}^{n} f_{r:n}(x)(1+D)^{n-r} = \sum_{r=1}^{n} \binom{n}{r} f_{r:r}(x) D^{r-1} \tag{2.8}$$

Example 3. By choosing the function $T(y) = y^k$, we find from (2.7) the identity

$$\sum_{r=1}^{n} f_{r:n}(x) \sum_{s=0}^{n-r} \binom{n-r}{s} k^{(s)} y^{k-s} = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) k^{(r-1)} y^{k-r+1}. \tag{2.9}$$

Comparing the coefficients of y^m on both sides of (2.9) yields the identity

$$\sum_{r=1}^{n} \binom{n-r}{k-m} f_{r:n}(x) = \binom{n}{k+1-m} f_{1:k+1-m}(x), \qquad \text{(Downton, 1966)}$$

2.2. Identities for Two Order statistics by Operator Methods:

(Balasubramanian *et al.*, 1992)

The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times \{1-F(y)\}^{n-s} f(x)f(y), \qquad -\infty < x < y < \infty \qquad (2.10)$$

Let Δ_1 and E_1 be the difference and the shift operators with common difference 1 acting on functions of w, and similarly Δ_2 and E_2 act on functions of z (where w and z are independent each being independent of both x and y). Then, we have the following operator equalities satisfied by joint distributions of two order statistics.

Theorem 3. For $n \ge 3$,

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)E_1^{r-1}E_2^{n-s}$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,r+1:s}(x,y)\Delta_1^{r-1}\Delta_2^{s-r-1}, \qquad (2.11)$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)E_1^{r-1}E_2^{s-r-1}$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,s:s}(x,y)\Delta_1^{r-1}\Delta_2^{s-r-1}, \qquad (2.12)$$

and

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)E_1^{s-r-1}E_2^{n-s}$$

$$= \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{1,r+1:s}(x,y) \Delta_1^{r-1} \Delta_2^{s-r-1}, \qquad (2.13)$$

Proof. Consider the identity

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} (1+u)^{r-1}(1+v)^{s-r-1}(1+w)^{n-s} f_{r,s:n}(x,y)$$

$$= \sum_{r=1}^{n-1} \frac{n!}{(r-1)!(n-r-1)!} \{(1+u)F(x)\}^{r-1}$$

$$\times \sum_{s=0}^{n-r-1} \binom{n-r-1}{s} [(1+v)\{F(y)-F(x)\}]^{s}$$

$$\times [(1+w)\{1-F(y)\}]^{n-r-1-s} f(x)f(y)$$

$$= n(n-1) \sum_{r=0}^{n-1} \binom{n-2}{r} \{(1+u)F(x)\}^{r}$$

$$\times [(1+v)\{F(y)-F(x)\} + (1+w)\{1-F(y)\}]^{n-r-2} f(x)f(y)$$

$$= n(n-1)[uF(x) + v\{F(y)-F(x)\} + w\{1-F(y)\} + 1]^{n-2} f(x)f(y). \qquad (2.14)$$

By setting $v=0$ and expanding the term on the right hand side of (2.11) binomially, we get

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} f_{r,s:n}(x,y)(1+u)^{r-1}(1+w)^{n-s}$$

$$= \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{r,s:s}(x,y) u^{r-1} w^{s-r-1}$$

from which the identity in (2.11) follows immediately by taking $u = \Delta_1$ and $w = \Delta_2$.By setting $w=0$ and $u=0$ in (2.14) and proceeding on similar lines, Balasubramanin *et al.*(1992) obtained the identities (2.12) and (2.13) respectively.

Example 4. Apply (2.11)-(2.13) to the function

$$T(w,z) = 1/(n-w-z-1).$$

Then setting $w = z = 0$ leads to the identities

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s-1}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)}f_{r,r+1:s}(x,y),$$

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{n-s+1}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)}f_{r,s:s}(x,y),$$

and

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{r}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)}f_{1,r+1:s}(x,y) \quad \text{respectively.}$$

These identities are due to Balakrishnan *et al.* (1992).

Example 5. Apply (2.11)-(2.13) to the function

$$T(w,z) = 1/(N-w-z-1)^{(k)}.$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-n+s-r)^{(k)}}f_{r,s:n}(x,y)$$

$$= \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{r,r+1:s}(x,y),$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-s-r)^{(k)}}f_{r,s:n}(x,y)$$

$$= \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{r,s:s}(x,y),$$

and

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-n+r)^{(k)}}f_{r,s:n}(x,y)$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{r,r+1:s}(x,y).$$

These three identities are generalizations of identities due to Balakrishnan *et al.*(1992) whose results correspond to the case $k=1$ and $N=n$.

Here is an equivalent form of Theorem 3 in terms of differentiation operator.

Theorem 4. For $n\geq 3$,

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}f_{r,s:n}(x,y)(1+D_1)^{r-1}(1+D_2)^{n-s}$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}f_{r,r+1:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \qquad (2.15)$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}f_{r,s:n}(x,y)(1+D_1)^{r-1}(1+D_2)^{s-r-1}$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}f_{r,s:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \qquad (2.16)$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}f_{r,s:n}(x,y)(1+D_1)^{s-r-1}(1+D_2)^{n-s}$$

$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}f_{1,r+1:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \qquad (2.17)$$

where D_1 and D_2 are differentiation operators acting on functions of w and z respectively.

Example 6. Let the operator identity (2.15) act on $T(w,z) = w^k z^l$. Then

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} f_{r,s:n}(x,y) \sum_{i=0}^{r-1} \sum_{j=0}^{n-s} \binom{r-1}{i} \binom{n-s}{j} k^{(i)} l^{(j)} w^{k-i} z^{l-j}$$

$$= \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{r,r+1:s}(x,y) k^{(r-1)} l^{(s-r-1)} w^{k-r+1} z^{l-s+r+1}. \quad (2.18)$$

comparing the coefficients of $w^{k-p} z^{l-q}$ on both sides of (2.18) gives the identity

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{r-1}{p} \binom{n-s}{q} f_{r,s:n}(x,y) = \binom{n}{p+q+2} f_{p+1,p+2:p+q+2}(x,y)$$

(Downtown, 1966)

In this chapter a unified method based on operators for establishing identities satisfied by order statistics is presented. Balasubramanian *et al.* (1992) have shown that many well-known identities for order statistics follow as special cases pf this method corresponding to different choices of operators and functions. They have also used this method to generate some new interesting identities satisfied by distributions of single as well as two order statistics. They made the following comments:

(i) In order to understand the usefulness of the operator method a few choices for the function T that the operator acts on is made. One may be able to generate many more identities for order statistics by making different choices for the function T.

(ii) For the purpose of illustration only the operators Δ and D are used. Other choices (*e.g.* matrices) are possible which could lead to many new identities for order statistics.

(iii) By writing the identities established in this chapter in terms of expectations of functions of order statistics and then by assuming specific distributions for the underlying population and making use of exact explicit expressions for these quantities, one can generate combinatorial identities. See, for example, Joshi & Balakrishnan (1981) and Balasubramanian & Beg (1990).

(iv) The operator method presented in this chapter, unlike the direct combinatorial method, lends itself to proving identities involving joint distributions of three or more order statistics in a straightforward manner.

3. Indicator Method for Recurrence Relation for Order Statistics for Non-identically Distributed Random Variables:

Balasubramanian & Balakrishnan (1992) have obtained the recurrence relations for order statistics using indicator method.

Let $X_1, X_2, \ldots, X_n$ be n random variables having an arbitrary joint distribution, with $X_{1:n}, X_{2:n}, \ldots X_{n:n}$ as the order statistics obtained by arranging these n variables. Now let $F_{r:n}(x_1)$ denote the distribution function of the order statistics $X_{r:n}$. Further, let $F^{[i]}_{r:n-1}(x_1)$ denote the distribution function of the order statistics $X^{[i]}_{r:n-1}$, where $X^{[i]}_{r:n-1}$ denotes the rth order statistics in $n-1$ variables obtained by dropping X_i from the original n variables.

For $-\infty < x_1 < \infty$, let us define the events A_i as

$A_i = \{X_i \le x_1\}$, for $i = 1,2,\ldots,n$. For any event A, let χ_A be the indicator variable taking on the value 1 when A occurs and zero otherwise.

By considering the special case when X_i's are independent and non-identically distributed. Balakrishnan (1988) and Bapat & Beg (1988) derived some recurrence relations satisfied by distributions of order statistics. Sathe & Dixit (1990) proved these relations for general case when the X_i's are arbitrarily distributed by using set theoretic arguments. These results were used by Balakrishnan *et al.* (1992) to establish some general relations and identities satisfied by order statistics arising from n arbitrarily distribute variables. In this chapter indicator functions of sets are used to prove these recurrence relations for order statistics. This approach, in addition to being simpler, lends itself to easy generalizations for higher orders.

3.1 Relations for single order statistics:

Here the 'triangle rule' (Arnold & Meeden, 1975, Arnold, 1977) for order statistics from arbitrary variables is proved in the following theorem.

Theorem 1. For $1 \le r \le n-1$ and $x_1 \in \Re$,

$$rF_{r+1:n}(x_1) + (n-1)F_{r:n}(x_1) = \sum_{i=1}^{n} F_{r:n-1}^{[i]} \tag{3.1}$$

Proof. Let us consider

$$\Phi = E\prod_{j=1}^{n}\{t_1\chi_{A_j} + t_2(1-\chi_{A_j})\} \tag{3.2}$$

$$= \sum_{r=0}^{n} t_1^r t_2^{n-r}\Pi_r, \tag{3.3}$$

where Π_r is the probability of exactly r of the A_j, s happening. It may be noted that for $r = 0,1,2\ldots,n$

$$\Pi_r = F_{r:n}(x_1) - F_{r+1:n}(x_1), \tag{3.4}$$

Or, equivalently,

$$F_{r:n}(x_1) = \sum_{j=r}^{n} \Pi_j, \tag{3.5}$$

with the convention that $F_{0:n}(x_1) \equiv 1$ and $F_{n+1:n}(x_1) \equiv 0$.

From (3.3), we obtain

$$\frac{\partial \Phi}{\partial t_1} + \frac{\partial \Phi}{\partial t_2} = \sum_{r=0}^{n} \left\{ r\, t_1^{r-1} t_2^{n-r} + (n-r) t_1^{r} t_2^{n-r-1} \right\} \Pi_r$$

$$= \sum_{r=0}^{n-1} t_1^{r} t_2^{n-r-1} \left\{ (r+1) \Pi_{r+1} (n-1) \Pi_r \right\} \tag{3.6}$$

Alternatively from (3.2) we directly find

$$\frac{\partial \Phi}{\partial t_1} + \frac{\partial \Phi}{\partial t_2} = \sum_{i=1}^{n} E \prod_{\substack{j=1 \\ j \neq i}}^{n} \left\{ t_1 \chi_{A_j} + t_2 (1 - \chi_{A_j}) \right\} \Pi_r$$

which upon using (3.3), yields

$$\frac{\partial \Phi}{\partial t_1} + \frac{\partial \Phi}{\partial t_2} = \sum_{i=1}^{n} \sum_{r=0}^{n-1} t_1^{r} t_2^{n-r-1} \Pi_r^{[i]}, \tag{3.7}$$

where

$$\Pi_r^{[i]} = F_{r:n-1}^{[i]}(x_1) - F_{r+1:n-1}^{[i]}(x_1)$$

Upon comparing the coefficients of $t_1^R t_2^{n-R-1}$ in (3.6) and (3.7), we find

$$(R+1)\Pi_{R+1}+(n-R)\Pi_R=\sum_{i=1}^{n}\Pi_R^{[i]},$$

which may be rewritten as

$$\{(R+1)\Pi_{R+1}-R\Pi_R\}+n\Pi_R=\sum_{i=1}^{n}\Pi_R^{[i]}. \tag{3.8}$$

The triangle rule in (3.1) follows from (3.8) by adding over R from r to n.

The indicator function method used in this chapter for deriving the recurrence relations for order statistics from arbitrarily variables can be suitably adopted to extend many other known recurrence relations and identities for order statistics in the independently identically distributed case to arbitrary case.

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